Dynamical centers for the elliptic quantum algebra $\mathcal{B}_{q,\lambda}(\hat{\mathfrak{g}})$

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The DVA

- DVA = deformation of Virasoro algebra; many contexts (RSOS symmetries, AGT duality in 5d, ZF algebras in XYZ spin chains...)
- Construction proposed by authors in Commun. Math. Phys. **354** (2017) 753, arXiv:1607.05050: bilinear traces in Lax operators of elliptic quantum algebras
- How about DYNAMICAL elliptic quantum algebras: which deformation of DVA may occur ?

General algebra $\mathcal{B}_{q,\lambda}(\hat{\mathfrak{g}})$

- Face-type/dynamical elliptic quantum algebra $\mathcal{B}_{q,\lambda}(\hat{\mathfrak{g}})$ = quasi-triangular quasi-Hopf algebra = Drinfeld twist on $\mathcal{U}_q(\hat{\mathfrak{g}})$, $\hat{\mathfrak{g}}$ affine Kac–Moody algebra
- Universal R-matrix $\mathcal{R}(\lambda)$ satisfies so-called *Gervais–Neveu–Felder* or *dynamical Yang–Baxter equation* :

$$\mathcal{R}_{12}(\lambda + h^{(3)})\,\mathcal{R}_{13}(\lambda)\,\mathcal{R}_{23}(\lambda + h^{(1)}) = \mathcal{R}_{23}(\lambda)\,\mathcal{R}_{13}(\lambda + h^{(2)})\,\mathcal{R}_{12}(\lambda)$$

- Evaluated R-matrix is: $R(z_1/z_2, \lambda) = \big(\pi_V(z_1) \otimes \pi_V(z_2)\big)\,\mathcal{R}(\lambda)$ hence RLL relations:

$$R_{12}(z_{12}, h)\, L_1^{\pm}(z_1)\, L_2^{\pm}(z_2, h^{(1)}) = L_2^{\pm}(z_2)\, L_1^{\pm}(z_1, h^{(2)})\, R_{12}(z_{12})$$
$$R_{12}(q^c z_{12}, h)\, L_1^{+}(z_1)\, L_2^{-}(z_2, h^{(1)}) = L_2^{-}(z_2)\, L_1^{+}(z_1, h^{(2)})\, R_{12}(q^{-c} z_{12})$$

R-matrix of $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)_c$ in the fundamental representation (Felder)

$$R(z,\lambda) = \rho(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & \bar{c}(z) & \bar{b}(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Elliptic functions in R matrix

$$b(z) = \frac{\Theta_p(q^2 w)\,\Theta_p(z)}{\Theta_p(w)\,\Theta_p(q^2 z)}\,, \qquad \bar{b}(z) = \frac{\Theta_p(q^2 w^{-1})\,\Theta_p(z)}{\Theta_p(w^{-1})\,\Theta_p(q^2 z)}\,,$$

$$c(z) = \frac{\Theta_p(q^2)\,\Theta_p(wz)}{\Theta_p(w)\,\Theta_p(q^2 z)}\,, \qquad \bar{c}(z) = \frac{\Theta_p(q^2)\,\Theta_p(w^{-1} z)}{\Theta_p(w^{-1})\,\Theta_p(q^2 z)}\,.$$

Here all parameters encapsulated as:
$\lambda = \rho + 1/2\, s\, \sigma_z + (r+2)d + \xi c$. Single dynamical parameter w is related to deformation parameter q as $w = q^{2s}$ and z is spectral parameter.
Jacobi Θ function: $\Theta_p(z) = (z;p)_\infty\,(pz^{-1};p)_\infty\,(p;p)_\infty$ with infinite multiple Pochhammer products:
$(z;p_1,\ldots,p_m)_\infty = \prod_{n_i \geq 0}(1 - z p_1^{n_1}\ldots p_m^{n_m})$.

Half Crossing relation

The dynamical elliptic R-matrices (R : Felder; $\widetilde{R}$: JKOS, twisted) satisfy the following crossing relations:

$$\sigma_y^{(1)} \left(R_{12}^{t_1}(z^{-1}q^{-2},\lambda)\right)^{-sl_1} \sigma_y^{(1)} \frac{\Upsilon(\lambda+\sigma_z^{(2)})}{\Upsilon(\lambda)} = R_{12}^{-1}(z^{-1},\lambda)$$

and

$$\sigma_y^{(1)} \Gamma_1 \left(\widetilde{R}_{12}^{t_1}(z^{-1}q^{-2},\lambda)\right)^{-sl_1} \Gamma_1(\sigma_z^{(2)})^{-1} \sigma_y^{(1)} \frac{\Upsilon(\lambda+\sigma_z^{(2)})}{\Upsilon(\lambda)} = \widetilde{R}_{12}^{-1}(z^{-1},\lambda)$$

where $\Upsilon(\lambda) = w^{-1/2}\,\Theta_p(w)$ and $\Gamma(\lambda) = (\det g)\,g(\lambda)^{-1}\,g(\lambda)^{-sc}$
g is additional, spectral-parameter-independent diagonal matrix twisting R to $\widetilde{R}$.

Full crossing relation

Crossing-unitarity relation (Jimbo Miwa Okado)

$$\left(\left(\widetilde{R}_{12}(z^{-1}q^{-4}, \lambda)^{-sl_2} \right)^{t_1} \right)^{-1} = \frac{1}{\mathfrak{n}(z)} \, G_1(\lambda)^{-1} \left(\widetilde{R}_{21}^{t_1}(z, \lambda) \right)^{-sc_2} G_1(\lambda - \sigma_z^{(2)})$$

where $G(\lambda) = \dfrac{\Upsilon(\lambda)}{\Upsilon(\lambda + \sigma_z)}$, can be deduced by double application of the half crossing relation.

Generating functions

Introduce the generating functions

$$t(z,\lambda) = \mathrm{Tr}\Big(N(\lambda)\, e^{-\sigma_z \partial}\, L^+(q^{-c}z,\lambda)^{-1}\, L^-(z,\lambda)\, e^{\sigma_z \partial}\Big) = \sum_{n\in\mathbb{Z}} t_n z^{-n}\,,$$

$$t^*(z,\lambda) = \mathrm{Tr}\Big(N(\lambda)\, e^{-\sigma_z \partial}\, L^-(q^{c}z,\lambda)^{-1}\, L^+(z,\lambda)\, e^{\sigma_z \partial}\Big) = \sum_{n\in\mathbb{Z}} t^*_n z^{-n}\,.$$

Diagonal matrix $N(\lambda)$ is given by

$$N(\lambda) = \Big(\frac{\Upsilon(\lambda)}{\Upsilon(\lambda+\sigma_z)}\Big)^{-sc} = \frac{\Upsilon(\lambda-\sigma_z)}{\Upsilon(\lambda)}$$

Exchange relations

They obey the exchange relations:

$$t(z_1,\lambda)\,L^{\pm}(z_2,\lambda) = L^{\pm}(z_2,\lambda)\,t(z_1,\lambda+\sigma_z) \qquad \text{when } c=-2$$
$$t^*(z_1,\lambda)\,L^{\pm}(z_2,\lambda) = L^{\pm}(z_2,\lambda)\,t^*(z_1,\lambda+\sigma_z) \qquad \text{when } c=2$$

$N(\lambda)$ diagonal $\Longrightarrow$ generating functionals $t(z_1,\lambda)$ and $t^*(z_1,\lambda)$ lie in $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)_c$. Not in $U_{q,p}(\widehat{gl}_2)$.
Dynamical shift on λ in exchange relations characterizes $t(z,\lambda)$ and $t^*(z,\lambda)$ as generating functionals for *dynamical* centers in $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)_c$.

Abelian algebra, quantum determinant ?

- At distinguished values of the central charge, $t(z,\lambda)$ (resp. $t^*(z,\lambda)$) realizes an abelian subalgebra of $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)_c$:

$$\big[t(z_1,\lambda)\,,\; t(z_2,\lambda)\big] = 0 \qquad \text{when } c = -2$$
$$\big[t^*(z_1,\lambda)\,,\; t^*(z_2,\lambda)\big] = 0 \qquad \text{when } c = 2.$$

- Traces also recast in terms of $U_{q,p}(\widehat{gl}_2)_c$ generators $\widetilde{L}^{\pm}(z) = L^{\pm}(z,\lambda)\, e^{\sigma_z \partial} \implies t^{(*)}(z_1)$ commute with $\widetilde{L}^{\pm}(z_2)$.
- Connection with quantum determinant ? Direct inspection indicates that they are independent of the quantum determinant. More later.
- Hence: genuine extended centers at the respective critical values similarly to the quantum affine and vertex elliptic cases.

Vertex-IRF on algebras

- The Vertex-IRF transformation is implemented by a matrix $S(z;p,w)$:

$$S_1(z_1;p,w)\,S_2(z_2;p,wq^{h_1})\,R^{IRF}(z_1/z_2;p,w) = R^{8V}((z_1/z_2)^{1/2};p)\,S_2(z_2;p,w)\,S_1(z_1;p,wq^{h_2})\,.$$

- Defines a morphism from $\mathcal{B}_{q,\lambda}(\widehat{gl}_2)_c$ to $\mathcal{A}_{q,p}(\widehat{gl}_2)_c$:

$$\phi \;:\; L^{IRF}(z) \;\to\; S(z;p,w)^{-1}\,L^{8V}(z^{1/2})\,S(z;p^*,wq^h)\,.$$

- Recall traces in $\mathcal{A}_{q,p}(\widehat{gl}_2)_c$:

$$t_{mn}(z) = \mathrm{Tr}\Big((g^{\frac12}hg^{\frac12})^{-m}\,L^{8V}((-p^{*\frac12})^n z)\,(g^{\frac12}hg^{\frac12})^{-n}\,L^{8V}(z)^{-1}\Big),$$

$$t^*_{-n,-m}(z) = \mathrm{Tr}\Big((g^{\frac12}hg^{\frac12})^{n}\,L^{8V}((-p^{\frac12})^{-m} z)^{-1}\,(g^{\frac12}hg^{\frac12})^{m}\,L^{8V}(z)\Big).$$

Vertex-IRF on traces

- Traces on non-dynamical side obey DVA-generating exchange relations:

$$\begin{aligned} t_{mn}(z_1)\, L^{8V}(z_2) &= \quad \mathcal{F}_{mn}(z_1/z_2)\, L^{8V}(z_2)\, t_{mn}(z_1), \\ L^{8V}(z_2)\, t^*_{-n,-m}(z_1) &= \mathcal{F}_{mn}(z_1/z_2)\, t^*_{-n,-m}(z_1)\, L^{8V}(z_2), \end{aligned}$$

- Original Lax matrix $L^{8V}(z)$ on surface $\mathscr{S}_{mn}$ defined by $(-p^{\frac{1}{2}})^m(-p^{*\frac{1}{2}})^n = q^{-2}$.
- Pullback $\phi^* \implies$ dynamical exchange algebra:

$$\begin{aligned} & s_{mn}(z_1;w)\, L^{IRF}(z_2) = \mathcal{F}_{mn}((z_1/z_2)^{1/2})\, L^{IRF}(z_2)\, \widetilde{s}_{mn}(z_1;w,h) \\ & s_{mn}(z;w) = S(z;p,w)^{-1}\, \phi^*\big(t_{mn}(z^{1/2})\big)\, S(z;p,w)\,, \\ & \widetilde{s}_{mn}(z;w,h) = S(z;p^*,wq^h)^{-1}\, \phi^*\big(t_{mn}(z^{1/2})\big)\, S(z;p^*,wq^h)\,. \end{aligned}$$

Vertex-IRF on traces, continued

- But $\phi^*\left(t_{mn}(z^{1/2})\right)$ may depend on w hence $\widetilde{s}_{mn}(z;w,h)$ not equal to $s_{mn}(z;wq^h)$.
- Notwithstanding this issue: Applying Vertex-IRF transformation to $t^*_{1,-1}(z)$ in $\mathcal{A}_{q,p}(\widehat{gl}_2)_c$: ratio of spectral parameters in the trace is reproduced correctly.
- Hence dynamical centers relate here to vertex-type surface $\mathscr{S}_{1,-1}$, $\implies$ consistent with no Liouville type formula since Liouville formula in elliptic case explicitely related to vertex-type surface $\mathscr{S}_{0,2}$.

why c=2 ?

- Critical level with extended center in affine algebra expected to be at $c = -2$. Why $c = 2$?
- Topological issue: Enveloping algebra requires local completion, vacuum representation and associated normal ordering for PBW procedure to be well-defined.
- Issue is that one can't have both t and t^* in SAME completion due to opposite orders of L^+ and L^- in two objects.
- hence $c = 2$ occurs all right, but in completion which has only lowest weight representations....
- Same issue with quantum affine algebras (vacuum representations go over to q-deformed case if q generic). Not so clear for elliptic or dynamical.

Perspectives

- Higher rank algebras ? $\mathcal{B}_{q,\lambda}(\widehat{gl}_N)_c$? Crossing-unitarity extends but subtle issue re. splitting of G function which follows from half-crossing; no half-crossing in $N \geq 3$?
- general affine Lie algebra ?
- Poisson bracket structures: expansion around $c = -2$?
- Critical Surfaces ? instead of "supercritical" points where exchange algebra degenerates. Equivalent of quasi periodicity property for non dynamical case?

HAPPY BIRTHDAY JEAN MICHEL