Dynamical centers for the elliptic quantum algebra $B_{q,\lambda}(\hat{g})$

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JMM 60, October 23rd, 2017
The DVA

- DVA = deformation of Virasoro algebra; many contexts (RSOS symmetries, AGT duality in 5d, ZF algebras in XYZ spin chains...)


- How about DYNAMICAL elliptic quantum algebras: which deformation of DVA may occur?
General algebra $\mathcal{B}_{q,\lambda}(\hat{g})$

- Face-type/dynamical elliptic quantum algebra $\mathcal{B}_{q,\lambda}(\hat{g}) = \text{quasi-triangular quasi-Hopf algebra} = \text{Drinfeld twist on } \mathcal{U}_q(\hat{g})$, $\hat{g}$ affine Kac–Moody algebra

- Universal $R$-matrix $\mathcal{R}(\lambda)$ satisfies so-called
  \textit{Gervais–Neveu–Felder} or \textit{dynamical Yang–Baxter equation}:

  $$\mathcal{R}_{12}(\lambda + h(3)) \mathcal{R}_{13}(\lambda) \mathcal{R}_{23}(\lambda + h(1)) = \mathcal{R}_{23}(\lambda) \mathcal{R}_{13}(\lambda + h(2)) \mathcal{R}_{12}(\lambda)$$

- Evaluated $R$-matrix is:
  $$R(z_1/z_2, \lambda) = (\pi_V(z_1) \otimes \pi_V(z_2)) \mathcal{R}(\lambda) \text{ hence RLL relations:}$$
  $$R_{12}(z_{12}, h) L_1^\pm(z_1) L_2^\pm(z_2, h^{(1)}) = L_2^\pm(z_2) L_1^\pm(z_1, h^{(2)}) R_{12}(z_{12})$$
  $$R_{12}(q^c z_{12}, h) L_1^+(z_1) L_2^-(z_2, h^{(1)}) = L_2^-(z_2) L_1^+(z_1, h^{(2)}) R_{12}(q^{-c} z_{12})$$
$R$-matrix of $B_{q,\lambda}(\hat{gl}_2)_c$ in the fundamental representation (Felder)

$$R(z, \lambda) = \rho(z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & \bar{c}(z) & \bar{b}(z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
Elliptic functions in R matrix

\[ b(z) = \frac{\Theta_p(q^2 w) \Theta_p(z)}{\Theta_p(w) \Theta_p(q^2 z)} , \quad \bar{b}(z) = \frac{\Theta_p(q^2 w^{-1}) \Theta_p(z)}{\Theta_p(w^{-1}) \Theta_p(q^2 z)} , \]
\[ c(z) = \frac{\Theta_p(q^2) \Theta_p(wz)}{\Theta_p(w) \Theta_p(q^2 z)} , \quad \bar{c}(z) = \frac{\Theta_p(q^2) \Theta_p(w^{-1} z)}{\Theta_p(w^{-1}) \Theta_p(q^2 z)} . \]

Here all parameters encapsulated as:
\[ \lambda = \rho + 1/2 s \sigma_z + (r + 2)d + \xi c. \]
Single dynamical parameter \( w \) is related to deformation parameter \( q \) as \( w = q^{2s} \) and \( z \) is spectral parameter.

Jacobi \( \Theta \) function: \( \Theta_p(z) = (z; p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty \) with infinite multiple Pochhammer products:
\[ (z; p_1, \ldots, p_m)_\infty = \prod_{n_i \geq 0} (1 - z p_1^{n_1} \cdots p_m^{n_m}) . \]
Half Crossing relation

The dynamical elliptic $R$-matrices ($R$: Felder; $\tilde{R}$: JKOS, twisted) satisfy the following crossing relations:

$$\sigma_y^{(1)} \left(R_{12}^{t_1}(z^{-1}q^{-2},\lambda)\right)^{-s_{h_1}} \sigma_y^{(1)} \frac{\gamma(\lambda + \sigma_z^{(2)})}{\gamma(\lambda)} = R_{12}^{-1}(z^{-1},\lambda)$$

and

$$\sigma_y^{(1)} \Gamma_1(\tilde{R}_{12}^{t_1}(z^{-1}q^{-2},\lambda))^{-s_{h_1}} \Gamma_1(\sigma_z^{(2)})^{-1} \sigma_y^{(1)} \frac{\gamma(\lambda + \sigma_z^{(2)})}{\gamma(\lambda)} = \tilde{R}_{12}^{-1}(z^{-1},\lambda)$$

where $\gamma(\lambda) = w^{-1/2} \Theta_p(w)$ and $\Gamma(\lambda) = (\det g) g(\lambda)^{-1} g(\lambda)^{-sc}$.

g is additional, spectral-parameter-independent diagonal matrix twisting $R$ to $\tilde{R}$. 

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Dynamical centers
Lyon 2017
Full crossing relation

Crossing-unitarity relation (Jimbo Miwa Okado)

\[
\left((-\tilde{R}_{12}(z^{-1}q^{-4},\lambda)^{-sl_2})^{t_1}\right)^{-1} = \frac{1}{n(z)} G_1(\lambda)^{-1} \left(\tilde{R}_{21}(z,\lambda)\right)^{-sc_2} G_1(\lambda - \sigma_z^{(2)})
\]

where \( G(\lambda) = \frac{\gamma(\lambda)}{\gamma(\lambda + \sigma_z)} \), can be deduced by double application of the half crossing relation.
Generating functions

Introduce the generating functions

\[
t(z, \lambda) = \text{Tr}\left( N(\lambda) e^{-\sigma_z \partial} L^+ (q^{-c} z, \lambda)^{-1} L^- (z, \lambda) e^{\sigma_z \partial} \right) = \sum_{n \in \mathbb{Z}} t_n z^{-n},
\]

\[
t^*(z, \lambda) = \text{Tr}\left( N(\lambda) e^{-\sigma_z \partial} L^- (q^c z, \lambda)^{-1} L^+ (z, \lambda) e^{\sigma_z \partial} \right) = \sum_{n \in \mathbb{Z}} t^*_n z^{-n}.
\]

Diagonal matrix \( N(\lambda) \) is given by

\[
N(\lambda) = \left( \frac{\Gamma(\lambda)}{\Gamma(\lambda + \sigma_z)} \right)^{-sc} = \frac{\Gamma(\lambda - \sigma_z)}{\Gamma(\lambda)}
\]
Exchange relations

They obey the exchange relations:

\[ t(z_1, \lambda) L^\pm(z_2, \lambda) = L^\pm(z_2, \lambda) t(z_1, \lambda + \sigma_z) \quad \text{when } c = -2 \]
\[ t^*(z_1, \lambda) L^\pm(z_2, \lambda) = L^\pm(z_2, \lambda) t^*(z_1, \lambda + \sigma_z) \quad \text{when } c = 2 \]

\[ N(\lambda) \text{ diagonal } \implies \text{generating functionals } t(z_1, \lambda) \text{ and } t^*(z_1, \lambda) \]
lie in \( B_{q,\lambda}(\hat{gl}_2)_c \). Not in \( U_{q,p}(\hat{gl}_2) \).

Dynamical shift on \( \lambda \) in exchange relations characterizes \( t(z, \lambda) \) and \( t^*(z, \lambda) \) as generating functionals for *dynamical* centers in \( B_{q,\lambda}(\hat{gl}_2)_c \).
Abelian algebra, quantum determinant?

- At distinguished values of the central charge, $t(z, \lambda)$ (resp. $t^*(z, \lambda)$) realizes an abelian subalgebra of $\mathcal{B}_{q,\lambda}(\hat{gl}_2)_c$:

  \[
  \begin{align*}
  [t(z_1, \lambda), t(z_2, \lambda)] &= 0 \quad \text{when } c = -2 \\
  [t^*(z_1, \lambda), t^*(z_2, \lambda)] &= 0 \quad \text{when } c = 2.
  \end{align*}
  \]

- Traces also recast in terms of $\mathcal{U}_{q,p}(\hat{gl}_2)_c$ generators

  \[
  \tilde{L}^\pm(z) = L^\pm(z, \lambda) e^{\sigma z \partial} \implies t^*(z_1) \text{ commute with } \tilde{L}^\pm(z_2).
  \]

- Connection with quantum determinant? Direct inspection indicates that they are independent of the quantum determinant. More later.

- Hence: genuine extended centers at the respective critical values similarly to the quantum affine and vertex elliptic cases.
Vertex-IRF on algebras

- The Vertex-IRF transformation is implemented by a matrix $S(z; p, w)$:

$$S_1(z_1; p, w) \cdot S_2(z_2; p, wq^{h_1}) \cdot R^{IRF}(z_1/z_2; p, w) = R^{V}((z_1/z_2)^{1/2}; p) \cdot S_2(z_2; p, w) \cdot S_1(z_1; p, wq^{h_2}).$$

- Defines a morphism from $\mathcal{B}_{q, \lambda}(\hat{gl}_2)_c$ to $\mathcal{A}_{q, p}(\hat{gl}_2)_c$:

$$\phi : L^{IRF}(z) \rightarrow S(z; p, w)^{-1} \cdot L^V(z^{1/2}) \cdot S(z; p^*, wq^h).$$

- Recall traces in $\mathcal{A}_{q, p}(\hat{gl}_2)_c$:

$$t_{mn}(z) = \text{Tr} \left( (g^{1/2} h g^{1/2})^{-m} L^V((-p^{1/2})^n z) \cdot (g^{1/2} h g^{1/2})^{-n} L^V(z)^{-1} \right),$$

$$t_{-n, -m}^*(z) = \text{Tr} \left( (g^{1/2} h g^{1/2})^n L^V((-p^{1/2})^{-m} z)^{-1} \cdot (g^{1/2} h g^{1/2})^m L^V(z) \right).$$
Vertex-IRF on traces

- Traces on non-dynamical side obey DVA-generating exchange relations:

\[ t_{mn}(z_1) L^8V(z_2) = F_{mn}(z_1/z_2) L^8V(z_2) t_{mn}(z_1), \]
\[ L^8V(z_2) t^*_{n,-m}(z_1) = F_{mn}(z_1/z_2) t^*_{n,-m}(z_1) L^8V(z_2), \]

- Original Lax matrix \( L^8V(z) \) on surface \( S_{mn} \) defined by

\[ (-p_1^{1/2})^m (-p_2^{1/2})^n = q^{-2}. \]

- Pullback \( \phi^* \implies \) dynamical exchange algebra:

\[ s_{mn}(z_1; w) L^{IRF}(z_2) = F_{mn}((z_1/z_2)^{1/2}) L^{IRF}(z_2) \tilde{s}_{mn}(z_1; w, h) \]
\[ s_{mn}(z; w) = S(z; p, w)^{-1} \phi^*(t_{mn}(z^{1/2})) S(z; p, w), \]
\[ \tilde{s}_{mn}(z; w, h) = S(z; p^*, wq^h)^{-1} \phi^*(t_{mn}(z^{1/2})) S(z; p^*, wq^h). \]
Vertex-IRF on traces, continued

- But $\phi^*(t_{mn}(z^{1/2}))$ may depend on $w$ hence $\tilde{s}_{mn}(z; w, h)$ not equal to $s_{mn}(z; wq^h)$.

- Notwithstanding this issue: Applying Vertex-IRF transformation to $t_{1,-1}^*(z)$ in $A_{q,p}(\hat{gl}_2)_c$: ratio of spectral parameters in the trace is reproduced correctly.

- Hence dynamical centers relate here to vertex-type surface $S_{1,-1}$, consistent with no Liouville type formula since Liouville formula in elliptic case explicitly related to vertex-type surface $S_{0,2}$.
why $c=2$?

- Critical level with extended center in affine algebra expected to be at $c = -2$. Why $c = 2$?
- Topological issue: Enveloping algebra requires local completion, vacuum representation and associated normal ordering for PBW procedure to be well-defined.
- Issue is that one can’t have both $t$ and $t^*$ in SAME completion due to opposite orders of $L^+$ and $L^-$ in two objects.
- hence $c = 2$ occurs all right, but in completion which has only lowest weight representations....
- Same issue with quantum affine algebras (vacuum representations go over to $q$-deformed case if $q$ generic). Not so clear for elliptic or dynamical.
Perspectives

- Higher rank algebras? $\mathcal{B}_{q,\lambda}(\hat{gl}_N)_c$? Crossing-unitarity extends but subtle issue re. splitting of $G$ function which follows from half-crossing; no half-crossing in $N \geq 3$?
- General affine Lie algebra?
- Poisson bracket structures: expansion around $c = -2$?
- Critical Surfaces? instead of "supercritical" points where exchange algebra degenerates. Equivalent of quasi periodicity property for non dynamical case?
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