Baxter Operator and Baxter Equation for Toda$_2$ and q-Toda chains.

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Plan of the talk

- 1– Toda_2 and q-Toda.
- 2– Bäcklund transformation.
- 3– Baxteriology.
- 4– Baxter operator and equation.
- 5– Solution of Baxter equation.

Toda$_1$ and Toda$_2$.

$$H = \sum_n \frac{1}{2} P_n^2 + Q_n^2, \quad Q_n = e^{\varphi_n - \varphi_{n+1}}$$

$$L(\mu) = \begin{pmatrix} -P_1 & Q_1 & 0 & \cdots & \cdots & \mu Q_N \\ Q_1 & -P_2 & Q_2 & 0 & \cdots & \cdots \\ 0 & Q_2 & -P_3 & Q_3 & 0 & \cdots \\ \vdots & & & & & \vdots \\ \mu^{-1} Q_N & 0 & \cdots & 0 & Q_{N-1} & -P_N \end{pmatrix}$$

One can check that $\mathrm{Tr} L^n(\mu)$ are in involution with respect to the first Poisson structure of the Toda chain

$$\begin{aligned} \{Q_n, Q_m\}_1 &= 0 \\ \{P_n, Q_m\}_1 &= Q_m(\delta_{nm} - \delta_{n-1,m}) \\ \{P_n, P_m\}_1 &= 0 \end{aligned}$$

$$P_n = \pi_n, \quad Q_n = e^{\varphi_n - \varphi_{n+1}}$$

$$\{\pi_n, \varphi_m\} = \delta_{nm}$$

However, it is well known that $\mathrm{Tr}L^n(\mu)$ are also in involution with respect to the second Poisson structure of the Toda chain M. Adler (1979).

$$\begin{aligned}
\{Q_n, Q_m\} &= Q_n Q_m(\delta_{n+1,m} - \delta_{n,m+1}) \\
\{Q_n, P_m\} &= -2Q_n P_m(\delta_{n,m} - \delta_{n+1,m}) \\
\{P_n, P_m\} &= -4Q_m^2\delta_{n,m+1} + 4Q_n^2\delta_{n+1,m}
\end{aligned}$$

In terms of canonical coordinates, one has

$$\begin{aligned}
Q_n^2 &= e^{-2\pi_{n+1}} e^{\varphi_n - \varphi_{n+1}} \\
P_n &= e^{-2\pi_n} + e^{\varphi_n - \varphi_{n+1}}
\end{aligned}$$

$$\{\pi_n, \varphi_m\} = \delta_{nm}$$

Why Toda$_2$

Consider

$$L(\mu)\xi = 0$$

Take the continuum limit with lattice spacing a

$$L(\mu)\xi = 0 \to a^2\left(\xi'' - \left(p' + p^2\right)\xi\right) = 0$$

where

$$p = \pi - \frac{1}{2}\varphi'$$

We should recover CFT in the thermodynamic limit of Toda$_2$. Cf. old works by Faddeev, Takhtadjan, Volkov, Kashaev, etc. on Liouville.

Quantum algebra.

The quantum version of the above Poisson bracket is easily extracted from these "old works".

$$
\begin{aligned}
Q_n Q_m &= q^{\frac{1}{2}(\delta_{n,m-1}-\delta_{n,m+1})} Q_m Q_n \\
P_n P_m &= P_m P_n + (q^{\frac{3}{2}} - q^{-\frac{1}{2}})(Q_n^2 \delta_{n+1,m} - Q_m^2 \delta_{n,m+1}) \\
P_n Q_m &= q^{(\delta_{n,m}-\delta_{n,m+1})} Q_m P_n
\end{aligned}
$$

Freidel-Maillet

It is better to work with the 2×2 Lax matrix

$$l_n(\lambda) = \begin{pmatrix} \lambda + P_n & -1 \\ Q_n^2 & 0 \end{pmatrix}$$

The model being non ultralocal, we look for relations of the type Freidel-Maillet (1991)

$$\begin{aligned} A_{12}(\lambda_1,\lambda_2) l_{1n}(\lambda_1) l_{2n}(\lambda_2) &= l_{2n}(\lambda_2) l_{1n}(\lambda_1) D_{12}(\lambda_1,\lambda_2) \\ l_{1n}(\lambda_1) l_{2,n+1}(\lambda_2) &= l_{2,n+1}(\lambda_2) C_{12}(\lambda_1,\lambda_2) l_{1n}(\lambda_1) \\ l_{2n}(\lambda_2) l_{1,n+1}(\lambda_1) &= l_{1,n+1}(\lambda_1) B_{12}(\lambda_1,\lambda_2) l_{2n}(\lambda_2) \end{aligned}$$

The monodromy matrix is given by

$$T_N(\lambda) = l_N(\lambda)\,\gamma(\lambda)\,l_{N-1}(\lambda)\,\gamma(\lambda)\,\cdots l_2(\lambda)\gamma(\lambda)\,l_1(\lambda)$$

where $\gamma(\lambda)$ denotes a numerical matrix satisfying

$$D_{12}\gamma_1 C_{12}\gamma_2 = \gamma_2 B_{12}\gamma_1 A_{12}, \quad \gamma(\lambda) = \begin{pmatrix} 1 & 0 \\ (1-q^2)\lambda & q^{-1/2} + (d_1/d_2)\lambda \end{pmatrix}$$

the parameter d_1/d_2 is arbitrary. Toda$_2$ corresponds to $d_1 = 0$.

Ultralocalization.

In terms of canonical coordinates, one has

$$\begin{aligned} Q_n^2 &= e^{-2\pi_{n+1}} e^{\varphi_n - \varphi_{n+1}} \\ P_n &= e^{-2\pi_n} + q^{-1/2} e^{\varphi_n - \varphi_{n+1}} \end{aligned}$$

with Weyl commutation relations

$$e^{-\varphi_n} e^{-2\pi_n} = q^2 e^{-2\pi_n} e^{-\varphi_n}$$

We write the quantum $l_n(\lambda)$ in terms of these canonical coordinates

$$l_n(\lambda) = \begin{pmatrix} \lambda + e^{-2\pi_n} + q^{-1/2} e^{\varphi_n - \varphi_{n+1}} & -1 \\ e^{-2\pi_n} e^{\varphi_n - \varphi_{n+1}} & 0 \end{pmatrix}$$

It is obviously not ultralocal.

However

$$\tilde{L}_n(\lambda) = G_{n+1}^{-1} l_n(\lambda) G_n = \begin{pmatrix} \lambda + e^{-2\pi_n} & [q^{-1/2}\lambda - (1 - q^{-1/2})e^{-2\pi_n}]e^{-\varphi_n} \\ e^{\varphi_n} & q^{-1/2} \end{pmatrix}$$

where $G_n = \begin{pmatrix} 1 & q^{-1/2}e^{2\pi_n} \\ 0 & e^{-2\pi_n}e^{-\varphi_n} \end{pmatrix}$ is already ultralocal. But the monodromy matrix involves the matrix $\gamma(\lambda)$. So there is one more step:

$$T_N(\lambda) = \cdots \gamma l_n(\lambda) \gamma l_{n-1}(\lambda) \gamma \cdots = \cdots \gamma G_{n+1} \tilde{L}_n(\lambda) G_n^{-1} \gamma(\lambda) G_n \tilde{L}_{n-1}(\lambda) \cdots$$

hence it is natural to define the ultralocal Lax matrix as

$$L_n(\lambda) = \tilde{L}_n(\lambda) G_n^{-1} \gamma(\lambda) G_n$$

We find

$$L_n(\lambda) = \begin{pmatrix} \lambda + e^{-2\pi_n} & \lambda[d_2 + d_1 e^{-2\pi_n}]e^{-\varphi_n} \\ e^{\varphi_n} & d_2 \end{pmatrix}$$

This ultralocal $L_n(\lambda)$ satisfies the usual Yang-Baxter equation

$$R_{12}(\lambda_1, \lambda_2) L_{1,n}(\lambda_1) L_{2,n}(\lambda_2) = L_{2,n}(\lambda_2) L_{1,n}(\lambda_1) R_{12}(\lambda_1, \lambda_2)$$

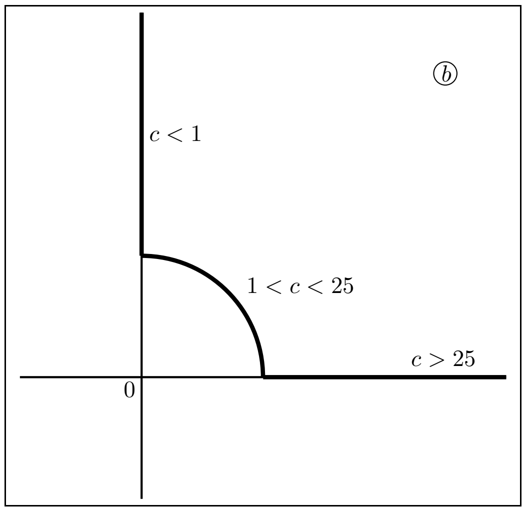
with quantum (twisted) R-matrix

$$R_{12}(\lambda_1, \lambda_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda_2 - \lambda_1}{q^2\lambda_2 - \lambda_1} & \frac{(q^2-1)\lambda_1}{q^2\lambda_2 - \lambda_1} & 0 \\ 0 & \frac{(q^2-1)\lambda_2}{q^2\lambda_2 - \lambda_1} & \frac{(\lambda_2 - \lambda_1)q^2}{q^2\lambda_2 - \lambda_1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Modular Invariance.

Since the R-matrix is independent of the lattice spacing Δ, it can be computed in the continuous CFT theory. The parameter q takes two values related to the central charge by :

$$c = 1 + 6\left(b + \frac{1}{b}\right)^2, \quad q = e^{i\pi b^2}, \quad \tilde{q} = e^{\frac{i\pi}{b^2}}$$



Faddeev modular symmetry

$$b^2 = \frac{\omega_1}{\omega_2}$$

The two dual commuting Weyl pairs

$$e^{-\omega_1\pi} e^{\frac{2\pi}{\omega_2}\varphi} = q^2 e^{\frac{2\pi}{\omega_2}\varphi} e^{-\omega_1\pi}$$

$$e^{-\omega_2\pi} e^{\frac{2\pi}{\omega_1}\varphi} = \tilde{q}^2 e^{\frac{2\pi}{\omega_1}\varphi} e^{-\omega_2\pi}$$

for $d_1 = 0$ we find Toda$_2$ (Harper). ($d_2 = e^{-\frac{2\pi}{\omega_2}\kappa_2}, \tilde{d}_2 = e^{-\frac{2\pi}{\omega_1}\kappa_2}$)

$$\begin{aligned} H_{\mathrm{Toda}_2} &= \sum_n e^{-\omega_1 \pi_n} + d_2 e^{\frac{2\pi}{\omega_2}(\varphi_n - \varphi_{n+1})} \\ \tilde{H}_{\mathrm{Toda}_2} &= \sum_n e^{-\omega_2 \pi_n} + \tilde{d}_2 e^{\frac{2\pi}{\omega_1}(\varphi_n - \varphi_{n+1})} \end{aligned}$$

For $d_2 = 0$ we find q-Toda. ($d_1 = e^{-\frac{2\pi}{\omega_2}\kappa_1}, \tilde{d}_1 = e^{-\frac{2\pi}{\omega_1}\kappa_1}$)

$$\begin{aligned} H_{q-\mathrm{Toda}} &= \sum_n \left\{1 + d_1 e^{\frac{2\pi}{\omega_2}(\varphi_{n-1} - \varphi_n)}\right\} e^{-\omega_1 \pi_n} \\ \tilde{H}_{q-\mathrm{Toda}} &= \sum_n \left\{1 + \tilde{d}_1 e^{\frac{2\pi}{\omega_1}(\varphi_{n-1} - \varphi_n)}\right\} e^{-\omega_2 \pi_n} \end{aligned}$$

We have

$$[H, \tilde{H}] = 0$$

and

$$\begin{cases} H^\dagger = H \\ \tilde{H}^\dagger = \tilde{H} \end{cases} \quad \text{if} \quad \omega_1, \omega_2 \quad \text{real}$$

$$H^\dagger = \tilde{H}, \quad \text{if} \quad \omega_2 = \omega_1^*$$

Our goal.

Let

$$t(\lambda) = \mathrm{Tr}\, L_N(\lambda) \cdots L_1(\lambda)$$

We want to construct Baxter $Q(y, x; \lambda)$ operator such that

- $Q_{\omega_1,\omega_2} = Q_{\omega_2,\omega_1}$ (modular invariance).
- $[Q(\lambda), Q(\mu] = 0$
- $[t(\lambda), Q(\mu)] = 0$
- $[\tilde{t}(\lambda), Q(\mu)] = 0$
- $t(\lambda)Q(\lambda) = a(\lambda)Q(q\lambda) + b(\lambda)Q(q^{-1}\lambda)$
- $\tilde{t}(\lambda)Q(\lambda) = \tilde{a}(\lambda)Q(\tilde{q}\lambda) + \tilde{b}(\lambda)Q(\tilde{q}^{-1}\lambda)$

Bäcklund tranformation.

To construct Baxter's Q operator, we will use its relation to Bäcklund transformations. The main observation of Gaudin, Pasquier 1992 is that Bäcklund transformations are related to the triangulation of the matrix $L_n(\lambda)$ by a gauge transformation, just as Baxter constructed his Q operator.

Bäcklund transformations are canonical transformations

$$(x_n, X_n) \to (y_n, Y_n), n = 1 \cdots N$$

that preserve the form of the Hamiltonians. This last property is achieved if the transformation acts on the Lax matrix by a gauge transformation i.e. there exist matrices $M_n(\lambda)$, depending eventually on the dynamical variables, such that (Kuztnetsov-Sklyanin, 1998)

$$L(\lambda; x_n, X_n)M_n(\lambda, x, y)) = M_{n+1}(\lambda, x, y)L(\lambda; y_n, Y_n) \qquad (*)$$

$$L(\lambda, x_n, X_n)M(\lambda, u_n, U_n) = M(\lambda, v_n, V_n)L(\lambda, y_n, Y_n)$$

Matrices L and M are symplectic orbits of Sklyanin bracket. This defines a symplectic transformation $(x_n, X_n, u_n, U_n) \to (y_n, Y_n, v_n, V_n)$. At the end impose $v_n = u_{n+1}, V_n = U_{n+1}$. As a result (for $d_2 = 0$, Bruschi, Ragnisco, 1988)

$$\begin{aligned} e^{-X_n} &= \frac{1 - d_1 e^{-(x_{n+1}-x_n)}}{1 - d_1 e^{-(x_n - x_{n-1})}} \frac{(t + e^{-(x_n - y_n)})(1 - d_2\, e^{-(y_{n+1}-x_n)})}{(1 + td_1\, e^{-(y_{n+1}-x_n)})} \\ e^{-Y_n} &= \frac{(t + e^{-(x_n - y_n)})(1 - d_2\, e^{-(y_n - x_{n-1})})}{(1 + td_1\, e^{-(y_n - x_{n-1})})} \end{aligned}$$

$$M(\lambda; u_n, U_n) = \begin{pmatrix} \lambda - te^{-U_n} & \lambda(1 - e^{-U_n})e^{-u_n} \\ e^{u_n} & 1 \end{pmatrix},$$

$$\begin{aligned} e^{-u_n} &= e^{-x_{n-1}} \\ e^{-U_n} &= -\frac{(1 - d_1 e^{-(x_n - x_{n-1})})(1 - d_2\, e^{-(y_n - x_{n-1})})}{(1 + d_1 t\, e^{-(y_n - x_{n-1})})} \end{aligned}$$

The relation with the triangulation of the Lax matrix, as needed to prove Baxter $T - Q$ equation, is as follows. Since $\det M_n(\lambda) = (\lambda - t)e^{-U_n}$ the matrix $M_n(t)$ is of rank one. The kernel is

$$M_n(t)\begin{pmatrix} 1 \\ -e^{x_{n-1}} \end{pmatrix} = 0$$

Then $L(t; x_n, X_n)M_n(t) = M_{n+1}(t)L(t; y_n, Y_n)$ implies

$$L_n(t; y_n, Y_n)\begin{pmatrix} 1 \\ -e^{x_{n-1}} \end{pmatrix} \propto \begin{pmatrix} 1 \\ -e^{x_n} \end{pmatrix}$$

as a consequence we have the triangulation property by a gauge transformation

$$\begin{pmatrix} 1 & 0 \\ e^{x_n} & 1 \end{pmatrix} L_n(t; y_n, Y_n) \begin{pmatrix} 1 & 0 \\ -e^{x_{n-1}} & 1 \end{pmatrix} = \begin{pmatrix} A_n & B_n \\ 0 & D_n \end{pmatrix}$$

$\mathbb{L}LM = ML\mathbb{L}$ - Operator.

The canonical transformation $(x_n, X_n, u_n, U_n) \to (y_n, Y_n, v_n, V_n)$ is replaced by a similarity transformation

$$(y, Y, v, V) = \mathbb{L}_t^{-1}(u,x)(x, X, u, U)\mathbb{L}_t(u,x)$$

Applying this to the equation

$$L(\lambda; x_n, X_n)M(\lambda; u_n, U_n) = M(\lambda; v_n, V_n)L(\lambda; y_n, Y_n)$$

we look for an operator $\mathbb{L}_t$, independent of λ, such that (Yang-Baxter !)

$$\mathbb{L}_t(u,x)\ L(\lambda; x, X)M_t(\lambda; u, U) = M_t(\lambda; u, U)L(\lambda; x, X)\ \mathbb{L}_t(u,x)$$

$$\mathbb{L}_t(u,x) = P_{xu}e^{-\alpha\omega_1 U}e^{-\frac{2\pi}{\omega_2}(u-x)\beta}\frac{S(-ae^{-\frac{2\pi}{\omega_2}t}(1-e^{-\omega_1 U})e^{-\frac{2\pi}{\omega_2}(u-x)})}{S(b(1-e^{-\omega_1 U})e^{-\frac{2\pi}{\omega_2}(u-x)})}$$

Here $a = q^2 d_2^{-1}, \quad b = q^2 d_1^{-1}$. This is a Volkov-type R matrix.

The function $S(x)$ is built from the quantum dilogarithm $\mathcal{S}(z)$

$$\frac{\mathcal{S}(z-i\omega_1)}{\mathcal{S}(z)} = \frac{1}{1-e^{-\frac{2\pi}{\omega_2}z}}, \quad \frac{\mathcal{S}(z-i\omega_2)}{\mathcal{S}(z)} = \frac{1}{1-e^{-\frac{2\pi}{\omega_1}z}}$$

It has an integral representation

$$\log \mathcal{S}(z) = \int_{\mathbb{R}+i0^+} \frac{dt}{t} \frac{e^{izt}}{(e^{\omega_1 t}-1)(e^{\omega_2 t}-1)}$$

The function $S(x)$ is related to to the quantum dilogarithm $\mathcal{S}(z)$ by

$$S(e^{-\frac{2\pi}{\omega_2}z}) = \mathcal{S}(z),$$

It satisfies

$$\frac{S(q^2x)}{S(x)} = \frac{1}{1-x}$$

$\mathbb{L}LM = ML\mathbb{L}$ - **Kernel.**

We look for $\mathbb{L}_t(u,x)$ as an integral operator represented by a kernel

$$\mathbb{L}_t(y,v;x,u) = \langle y| \otimes \langle v| \, \mathbb{L}_t \, |x\rangle \otimes |u\rangle$$

where $|x\rangle$ is the basis where the operator x acts by multiplication (and similarly for u).

$$e^{-\frac{2\pi}{\omega_2}x}|x'\rangle = e^{-\frac{2\pi}{\omega_2}x'}|x'\rangle, \quad e^{-\omega_1 X}|x'\rangle = |x' - i\omega_1\rangle,$$

$$\mathbb{L}_t(y,v;x,u) = \delta(x-v)e^{-\frac{2i\pi}{\omega_1\omega_2}t(x-y)} \frac{\mathcal{S}(y-u+\kappa_2-\frac{i}{2}\Omega)\mathcal{S}(x-u+\kappa_1+\frac{i}{2}\Omega)}{\mathcal{S}(x-y-t+\frac{i}{2}\Omega)\mathcal{S}(y-u+\kappa_1+t)}$$

Here $\Omega = \omega_1 + \omega_2$. Modular invariance is explicit.

More Baxteriology.

In exactly the same way, we can find an operator $\mathbb{R}_{tt'}(u,v)$ such that

$$\mathbb{R}_{tt'}(u,v)M_t(u)M_{t'}(v) = M_{t'}(v)M_t(u)\mathbb{R}_{tt'}(u,v)$$

We find

$$\mathbb{R}_{tt'} = P_{uv}\frac{S(-q^2\hat{t}'(1-e^{-\omega_1 V})e^{-\frac{2\pi}{\omega_2}(v-u)})}{S(-q^2\hat{t}(1-e^{-\omega_1 V})e^{-\frac{2\pi}{\omega_2}(v-u)})}, \quad \hat{t} = e^{-\frac{2\pi}{\omega_2}t}$$

Combined with $\mathbb{L}LM = ML\mathbb{L}$ we can write the transformation

$$L(\lambda;x,X)M_t(\lambda;u,U)M_{t'}(\lambda;v,V) \to M_{t'}(\lambda;v,V)M_t(\lambda;u,U)L(\lambda;x,X)$$

in two different ways and as usual we expect the compatibility relation

$$\mathbb{R}_{t,t'}(u,v)\mathbb{L}_{t'}(v,x)\mathbb{L}_t(u,x) = \mathbb{L}_t(u,x)\mathbb{L}_{t'}(v,x)\mathbb{R}_{t,t'}(u,v)$$

It follows essentially from Schützenberger relation (Volkov 2003)

$$S(e^x + e^X) = S(e^x)S(e^X)$$

or pentagon identity (Kashaev, 2015.)

Baxter Q operator.

Baxter Q-operator is defined as

$$Q_t = \mathrm{Tr}_{V_u} \, \mathbb{L}_N(t) \cdots \mathbb{L}_1(t)$$

in exact parallel to the transfer matrix

$$t(\lambda) = \mathrm{Tr}_{V_{\frac{1}{2}}} \, L_N(\lambda) \cdots L_1(\lambda)$$

One has the properties

- $[t(\lambda), t(\lambda')] = 0$ follows from $R(\lambda, \lambda')L(\lambda)L(\lambda') = L(\lambda')L(\lambda)R(\lambda, \lambda')$
- $[t(\lambda), Q_t] = 0$ follows from $M(\lambda, t)L(\lambda)\mathbb{L}(t) = \mathbb{L}(t)L(\lambda)M(\lambda, t)$
- $[Q_t, Q_{t'}] = 0$ follows from $\mathbb{R}(t, t')\mathbb{L}(t')\mathbb{L}(t) = \mathbb{L}(t)\mathbb{L}(t')\mathbb{R}(t, t')$

and dual equations.

Using the kernel representation for $\mathbb{L}_\lambda$, and remembering that $x_n = v_n = u_{n+1}$, we represent the operator Q_λ by a kernel

$$Q_\lambda(y,x) = \int du_N \cdots \int du_1 \prod_{j=1}^{N} \mathbb{L}_\lambda(y_j, u_{j+1}; x_j, u_j)$$

The integrals can be done because of the δ-functions.
We obtain

$$Q_\lambda(y,x) = \prod_{j=1}^{N} w_j(y_j, x_j, x_{j-1}; \lambda)$$

with

$$w_j = e^{-\frac{2i\pi}{\omega_1\omega_2}\lambda(x_j - y_j)} \frac{\mathcal{S}(y_j - x_{j-1} + \kappa_2 - \frac{i}{2}\Omega)\mathcal{S}(x_j - x_{j-1} + \kappa_1 + \frac{i}{2}\Omega)}{\mathcal{S}(x_j - y_j - \lambda + \frac{i}{2}\Omega)\mathcal{S}(y_j - x_{j-1} + \kappa_1 + \lambda)}$$

Baxter equation.

We now derive the Baxter equation. We have

$$\begin{aligned} Q(y,x;\lambda) &= w_N(y_N,x,\lambda)\cdots w_1(y_1,x,\lambda) \\ t(\lambda,y) &= \mathrm{Tr}_{V_{\frac{1}{2}}}\ L_N(y_N,Y_N)\cdots L_1(y_1,Y_1) \end{aligned}$$

hence $t(\lambda,y)Q_\lambda(y,x) = \mathrm{Tr}_{V_{\frac{1}{2}}}\ (L_N(\lambda;y_N,Y_N)w_N)\cdots(L_1(\lambda;y_1,Y_1)w_1)$
where $(L_j(\lambda;y_j,Y_j)w_j)$ means

$$L(\lambda;y_j,Y_j)w_j(y_j;x_{j-1},x_j;\lambda) = \begin{pmatrix} (e^{-\frac{2\pi}{\omega_2}\lambda} - qe^{-\omega_1 Y_j})w_j & e^{-\frac{2\pi}{\omega_2}\lambda}(d_2 + qd_1 e^{-\omega_1 Y_j})e^{-\frac{2\pi}{\omega_2}y_j}w_j \\ e^{\frac{2\pi}{\omega_2}y_j}w_j & d_2 w_j \end{pmatrix}$$

At this point we recover our triangulation property

$$L(y_j,Y_j)w_j(y_j;x_{j-1},x_j) = \begin{pmatrix} 1 & 0 \\ e^{\frac{2\pi}{\omega_2}x_j} & 1 \end{pmatrix} \begin{pmatrix} A_j & B_j \\ 0 & D_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{\frac{2\pi}{\omega_2}x_{j-1}} & 1 \end{pmatrix}$$

hence,

$$t(\lambda)Q(\lambda) = \prod_{j=1}^{N} A_j + \prod_{j=1}^{N} D_j$$

From this we get immediately Baxter equation.

$$t(\lambda)Q_\lambda(y,x) = (-q^{-1}e^{\frac{2\pi}{\omega_2}\lambda})^N e^{-\omega_1 p_0} Q_{\lambda-i\omega_1}(y,x) + e^{\frac{2N\pi}{\omega_2}\lambda}(d_2+q^{-2}d_1 e^{-\frac{2\pi}{\omega_2}\lambda})^N Q_{\lambda+i\omega_1}(y,x)$$

Applying to an eigenvector $v(x)$ of both $t(\lambda)$ and $Q_\lambda(y,x)$

$$\int dx Q_\lambda(y,x)v(x) = q(\lambda)v(y)$$

we obtain the scalar Baxter equation

$$t_\tau(\lambda)q(\lambda) = \rho^{\frac{\omega_1}{2}}(\sigma q(\lambda - i\omega_1) + \sigma^{-1}q(\lambda + i\omega_1))$$

- q-Toda

$$t_\tau(\lambda) = \prod 2\sinh\frac{1}{\omega_2}(\lambda - \tau_k), \quad \sigma = i^{-N}, \quad \rho = e^{-\frac{2\pi N}{\omega_1\omega_2}\kappa_1}$$

- Toda$_2$

$$t_\tau(\lambda) = \prod_k (e^{-\frac{2\pi}{\omega_2}\lambda} - e^{-\frac{2\pi}{\omega_2}\tau_k}), \quad \sigma = (-1)^N, \quad \rho = e^{-\frac{2\pi N}{\omega_1\omega_2}\kappa_2}e^{-p_0}$$

We of course have the dual equations $\omega_1 \leftrightarrow \omega_2$

Analyticity properties of the solution

$$\int dx Q_\lambda(y,x)v(x) = q(\lambda)v(y)$$

$$Q_\lambda(y,x) = \prod_{j=1}^{N} w_j(y_j, x_j, x_{j-1}; \lambda)$$

$$w_j = e^{-\frac{2i\pi}{\omega_1\omega_2}\lambda(x_j-y_j)} \frac{\mathcal{S}(y_j - x_{j-1} + \kappa_2 - \frac{i}{2}\Omega)\mathcal{S}(x_j - x_{j-1} + \kappa_1 + \frac{i}{2}\Omega)}{\mathcal{S}(x_j - y_j - \lambda + \frac{i}{2}\Omega)\mathcal{S}(y_j - x_{j-1} + \kappa_1 + \lambda)}$$

According to Bytsko-Teschner, $q(\lambda)$ develops a pole when there is a pinching of the integration contour by poles of Q for values of λ independent of x, y. This can happen only between the first term in the numerator and the second term in the denominator. For q-Toda the first term is absent and there is no pinching. For Toda$_2$ the second term is absent and there is no pinching either. Hence

For both q-Toda and Toda$_2$, $q(\lambda)$ is an *entire function of* λ*.*

Solution of Baxter equation.

We want to solve Baxter equations with $q(\lambda)$ entire.

$$\begin{aligned} t_\tau(\lambda)q(\lambda) &= \rho(\sigma q(\lambda - i\omega_1) + \sigma^{-1}q(\lambda + i\omega_1) \\ \tilde{t}_{\tilde{\tau}}(\lambda)q(\lambda) &= \rho(\sigma q(\lambda - i\omega_2) + \sigma^{-1}q(\lambda + i\omega_2) \end{aligned}$$

We introduce open chains Baxter equations.

$$t_\delta(\lambda)q_L(\lambda) = \rho\sigma q_L(\lambda - i\omega_1), \quad t_\delta(\lambda)q_R(\lambda) = \rho\sigma^{-1}q_R(\lambda + i\omega_1)$$

solutions are

$$q_L(\lambda) = f_L(\lambda)\prod_k \mathcal{S}^{-1}(\lambda - \delta_k), \quad q_R(\lambda) = f_R(\lambda)\prod_k \mathcal{S}^{-1}(-\lambda + \delta_k)$$

where f_R, f_L are Gaussian functions. These functions are *modular invariant*. We promote them to *exact meromorphic* solutions of Baxter equation for the closed chain by multiplying them by correction factors

$$\begin{aligned} q_+(\lambda) &= q_L(\lambda)\nu_\uparrow(\lambda)\tilde{\nu}_\uparrow(\lambda) \\ q_-(\lambda) &= q_R(\lambda)\nu_\downarrow(\lambda - i\omega_1)\tilde{\nu}_\downarrow(\lambda - i\omega_2) \end{aligned}$$

Plugging into Baxter equation, and using $i\omega_2$-periodicity of $\nu_{\uparrow/\downarrow}$ and $i\omega_1$-periodicity of $\tilde{\nu}_{\uparrow/\downarrow}$:

$$\nu_{\uparrow/\downarrow}(\lambda + i\omega_2) = \nu_{\uparrow/\downarrow}(\lambda), \quad \tilde{\nu}_{\uparrow/\downarrow}(\lambda + i\omega_1) = \tilde{\nu}_{\uparrow/\downarrow}(\lambda)$$

we find

$$\begin{aligned} t_\delta(\lambda)\nu_\uparrow(\lambda - i\omega_1) + \frac{\rho^{\omega_1}}{t_\delta(\lambda + i\omega_1)}\nu_\uparrow(\lambda + i\omega_1) &= t_\tau(\lambda)\nu_\uparrow(\lambda) \\ t_\delta(\lambda)\nu_\downarrow(\lambda) + \frac{\rho^{\omega_1}}{t_\delta(\lambda - i\omega_1)}\nu_\downarrow(\lambda - 2i\omega_1) &= t_\tau(\lambda)\nu_\downarrow(\lambda - i\omega_1) \end{aligned}$$

$$\begin{aligned} \tilde{t}_\delta(\lambda)\tilde{\nu}_\uparrow(\lambda - i\omega_2) + \frac{\rho^{\omega_2}}{\tilde{t}_\delta(\lambda + i\omega_2)}\tilde{\nu}_\uparrow(\lambda + i\omega_2) &= \tilde{t}_{\tilde{\tau}}(\lambda)\tilde{\nu}_\uparrow(\lambda) \\ \tilde{t}_\delta(\lambda)\tilde{\nu}_\downarrow(\lambda) + \frac{\rho^{\omega_2}}{\tilde{t}_\delta(\lambda - i\omega_2)}\tilde{\nu}_\downarrow(\lambda - 2i\omega_2) &= \tilde{t}_{\tilde{\tau}}(\lambda)\tilde{\nu}_\downarrow(\lambda - i\omega_2) \end{aligned}$$

We can eliminate $t_\tau(\lambda)$. We find the relation

$$K(\lambda - i\omega_1) = K(\lambda)$$

where $K(\lambda)$ is the Wronskian

$$K(\lambda) = \nu_\uparrow(\lambda)\nu_\downarrow(\lambda) - \frac{\rho^{\omega_1}}{t_\delta(\lambda)t_\delta(\lambda + i\omega_1)}\nu_\uparrow(\lambda + i\omega_1)\nu_\downarrow(\lambda - i\omega_1)$$

With $i\omega_2$-periodicity of $\nu_{\uparrow/\downarrow}$, we also have

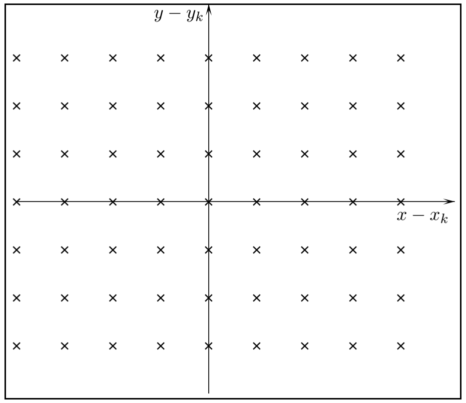
$$K(\lambda - i\omega_2) = K(\lambda)$$

hence $K(\lambda)$ is an elliptic function. We make the choice $K(\lambda) = 1$. This amounts to fixing the arbitrary elliptic function multiplying the solutions of Baxter equation. We get

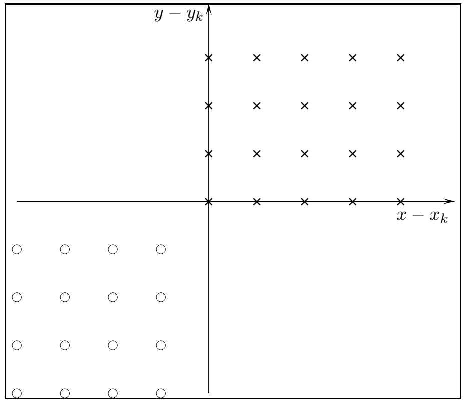
$$\nu_\uparrow(\lambda)\nu_\downarrow(\lambda) = 1 + \frac{\rho^{\omega_1}}{t_\delta(\lambda)t_\delta(\lambda + i\omega_1)}\nu_\uparrow(\lambda + i\omega_1)\nu_\downarrow(\lambda - i\omega_1)$$

and of course the dual equation also holds

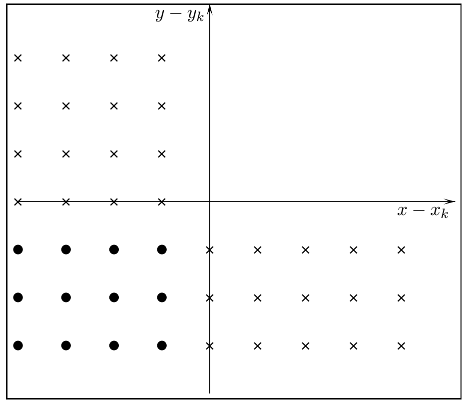
$$q_+(\lambda) \quad = \quad q_L(\lambda) \quad \times \quad \nu_\uparrow(\lambda)\tilde{\nu}_\uparrow(\lambda)$$



Pole lattice of the functions $q_+(\lambda)$ and $q_-(\lambda)$.

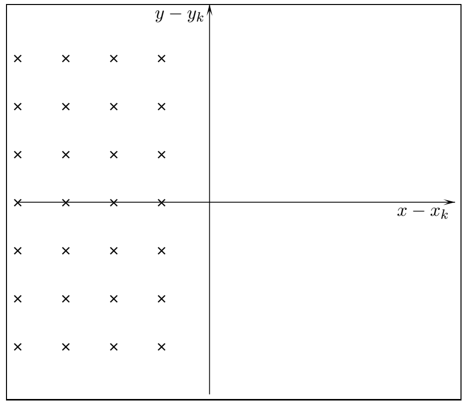


Poles and zeroes of $q_L(\lambda)$. $\lambda - \delta_k = i\omega_1(x - x_k) + i\omega_2(y - y_k)$.



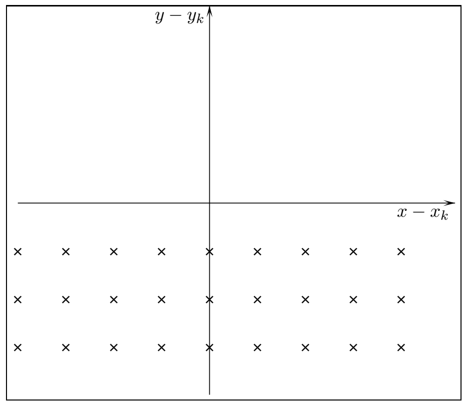
Pole pattern of $\nu_\uparrow(\lambda)\tilde{\nu}_\uparrow(\lambda)$. $\times$ simple pole, $\bullet$ double pole.

$\nu_\uparrow(\lambda)$



Pole pattern of $\nu_\uparrow(\lambda)$ using $i\omega_2$ periodicity.

$\tilde{\nu}_\uparrow(\lambda)$



Pole pattern of $\tilde{\nu}_\uparrow(\lambda)$ using $i\omega_1$ periodicity.

Solution by NLIE.

$$\nu_\uparrow(\lambda)\nu_\downarrow(\lambda) = 1 + \frac{\rho^{\omega_1}}{t_\delta(\lambda)t_\delta(\lambda + i\omega_1)}\nu_\uparrow(\lambda + i\omega_1)\nu_\downarrow(\lambda - i\omega_1)$$

is a factorization problem ! Follow Nekrasov-Shatashvili, 2009; Kozlowski-Teschner, 2010 : Define Y_δ solution of the NLIE

$$\begin{aligned}\log Y_\delta(\lambda) \quad &= \quad \int_{\mathcal{C}} \frac{d\tau}{2i\omega_2}\left\{\coth\left[\frac{\pi}{\omega_2}(\lambda - \tau - i\omega_1)\right] - \coth\left[\frac{\pi}{\omega_2}(\lambda - \tau + i\omega_1)\right]\right\} \times \\ &\times \log\left[1 + \frac{\rho^{\omega_1}Y_\delta(\tau)}{\mathbf{t}_\delta(\tau - i\omega_1/2)\mathbf{t}_\delta(\tau + i\omega_1/2)}\right]\end{aligned}$$

Then

$$\begin{aligned}v_\uparrow(\lambda) \quad &= \quad \exp\left\{-\int_{\mathcal{C}} \frac{d\tau}{2i\omega_2}\left(\coth\left[\frac{\pi}{\omega_2}(\lambda - \tau + \frac{i\omega_1}{2})\right] + 1\right)\log \mathcal{V}_\delta(\tau)\right\}, \quad \lambda \in \mathcal{B}_+ - \\ v_\downarrow(\lambda - i\omega_1) \quad &= \quad \exp\left\{\int_{\mathcal{C}} \frac{d\tau}{2i\omega_2}\left(\coth\left[\frac{\pi}{\omega_2}(\lambda - \tau - \frac{i\omega_1}{2})\right] + 1\right)\log \mathcal{V}_\delta(\tau)\right\}, \quad \lambda \in \mathcal{B}_- + i\end{aligned}$$

where $$\mathcal{V}_\delta(\tau) = 1 + \frac{\rho^{\omega_1}Y_\delta(\tau)}{\mathbf{t}_\delta(\tau - i\omega_1/2)\mathbf{t}_\delta(\tau + i\omega_1/2)}$$

Bethe equations.

$$q(\lambda) = q_+(\lambda) - \xi q_-(\lambda)$$

To kill the poles, which are on the lattice $\delta_k + i\omega_1\mathbb{Z} + i\omega_2\mathbb{Z}$, we have to impose only N Bethe equations

- q-Toda

$$\frac{2Ni\pi\kappa_1}{\omega_1\omega_2}\delta_k - \log\xi - \frac{N\pi}{\omega_1\omega_2}\Omega\delta_k + \sum_{l\neq k}\log\frac{\mathcal{S}(-\delta_k+\delta_l)}{\mathcal{S}(\delta_k-\delta_l)} + I(\delta_k) + \tilde{I}(\delta_k) = i\pi(2n_k+1)$$

- Toda$_2$

$$\frac{2Ni\pi\kappa_2}{\omega_1\omega_2}\delta_k - \log\xi - \frac{N\pi}{\omega_1\omega_2}\Omega\delta_k - \frac{Ni\pi}{\omega_1\omega_2}\delta_k^2 + \sum_{l\neq k}\log\frac{\mathcal{S}(-\delta_k+\delta_l)}{\mathcal{S}(\delta_k-\delta_l)} + I(\delta_k) + \tilde{I}(\delta_k) = i\pi(2n_k+1)$$

$$\frac{\nu_\uparrow(\lambda)}{\nu_\downarrow(\lambda - i\omega_1)} = e^{I(\lambda)}, \qquad \frac{\tilde{\nu}_\uparrow(\lambda)}{\tilde{\nu}_\downarrow(\lambda - i\omega_2)} = e^{\tilde{I}(\lambda)}$$

Notice that these Bethe equations are modular invariant.

Conclusion

Bon Anniversaire
Jean-Michel