The free boundary Schur process and applications

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Joint work with Dan Betea, Peter Nejjar and Mirjana Vuletić
based on arXiv:1704.05809

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Introduction

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- last-passage percolation and exclusion processes.

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There exists several variants of the Schur process (pfaffian, periodic, shifted...) and here we consider the case of free (open) boundary conditions.

Outline

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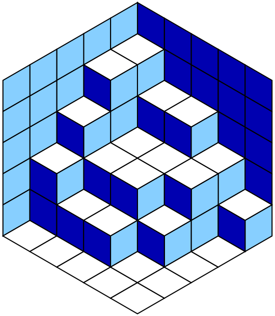
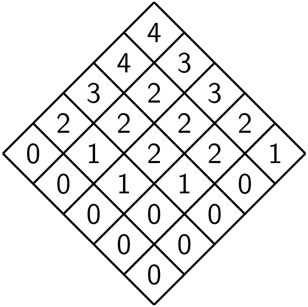
Plane partitions and Schur processes [Okounkov-Reshetikhin 2003]

Lozenge tiling



Plane partition



Sequence of interlaced integer partitions

$$\emptyset \prec 2 \prec \begin{matrix}3\\1\end{matrix} \prec \begin{matrix}4\\2\\1\end{matrix} \prec \begin{matrix}4\\2\\2\end{matrix} \succ \begin{matrix}3\\2\\1\end{matrix} \succ \begin{matrix}3\\2\end{matrix} \succ 2 \succ 1$$

Partitions, interlacing, Schur functions

An (integer) partition λ is a nonincreasing sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

that vanishes eventually. Its size is $|\lambda| := \sum \lambda_i$.

Two partitions λ, μ are said interlaced, which we write $\lambda \succ \mu$, iff

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots$$

Such constraint can be implemented via skew Schur functions of a single variable:

$$s_{\lambda/\mu}(q) = q^{|\lambda|-|\mu|} \mathbb{1}_{\lambda \succ \mu}.$$

Plane partitions and Schur processes [Okounkov-Reshetikhin 2003]

Consider a sequence $\underline{\lambda} = \cdots, \lambda^{(-2)}, \lambda^{(-1)}, \lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \cdots$ of integer partitions with finite support, and set

$$W(\underline{\lambda}) = \cdots s_{\lambda^{(-1)}/\lambda^{(-2)}}(q^{3/2}) s_{\lambda^{(0)}/\lambda^{(-1)}}(q^{1/2}) s_{\lambda^{(0)}/\lambda^{(1)}}(q^{1/2}) s_{\lambda^{(1)}/\lambda^{(2)}}(q^{3/2}) \cdots$$

Proposition

The weight $W(\underline{\lambda})$ is nonzero iff $\underline{\lambda}$ corresponds to a plane partition π, in which case $W(\underline{\lambda}) = q^{\mathrm{vol}(\pi)}$.
Plane partitions whose shape fits in a $N \times N$ square correspond to sequences vanishing outside the interval $[-N, N]$.

Plane partitions and Schur processes [Okounkov-Reshetikhin 2003]

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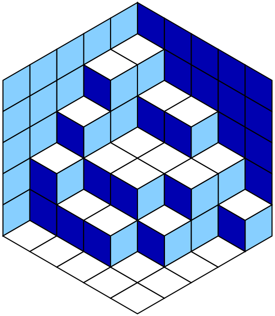

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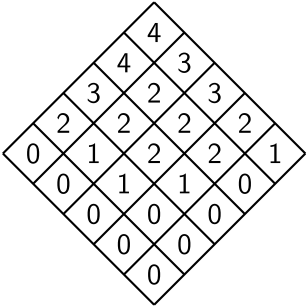

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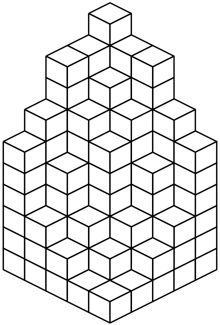
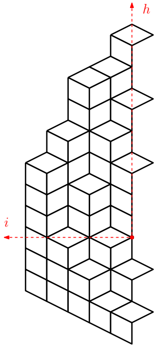
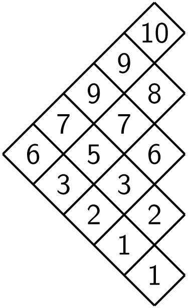
- By changing the order of some interlacings in $W(\underline{\lambda})$, one can treat "skew plane partitions" [Okounkov-Reshetikhin 2007].
- The form of $W(\underline{\lambda})$ is suitable for the transfer matrix method.

Our interest here: symmetric/free boundary tilings







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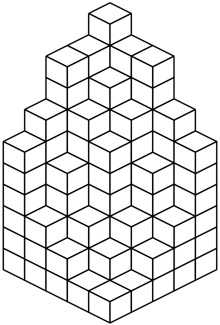

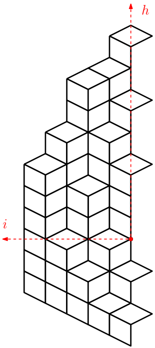

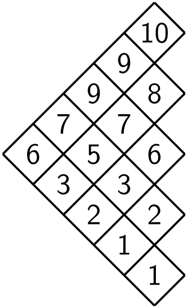

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One free boundary (or pfaffian) Schur process

Consider a sequence $\underline{\lambda} = \lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \cdots$ of integer partitions with finite support, and set

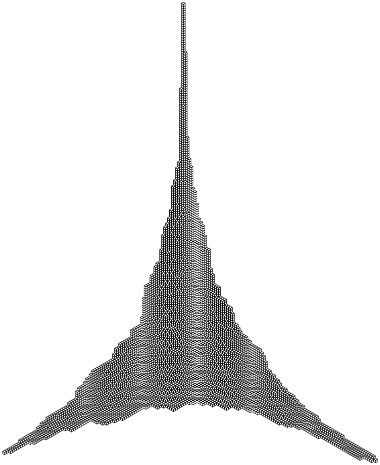
$$W(\underline{\lambda}) = s_{\lambda^{(0)}/\lambda^{(1)}}(q^{1/2}) s_{\lambda^{(1)}/\lambda^{(2)}}(q^{3/2}) \cdots$$

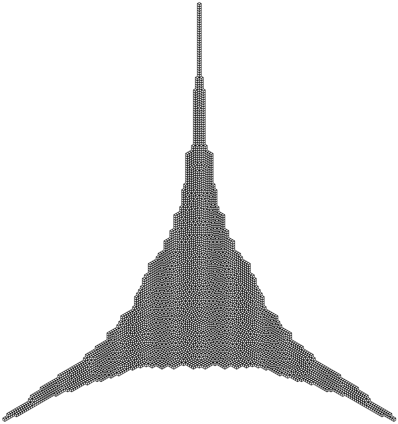
Proposition

The weight $W(\underline{\lambda})$ is nonzero iff $\underline{\lambda}$ corresponds to a vertically symmetric plane partition π, in which case $W(\underline{\lambda}) = q^{\mathrm{vol}(\pi)/2}$.

- This is an instance of pfaffian Schur process [Borodin-Rains 2005, see also Sasamoto-Imamura 2003].

Large objects ($q \to 1$): limit shape





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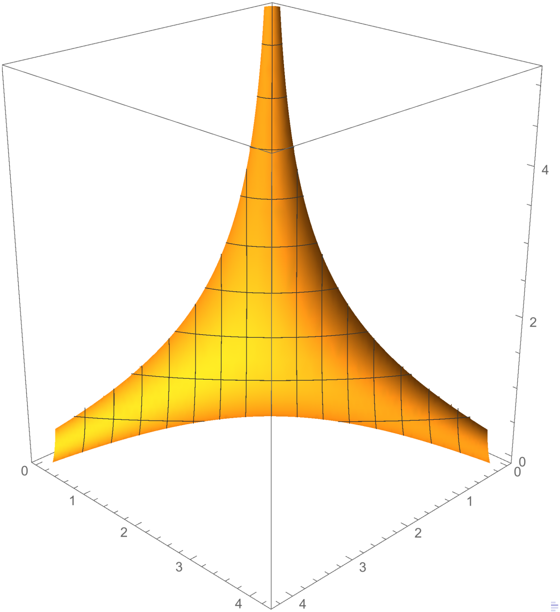
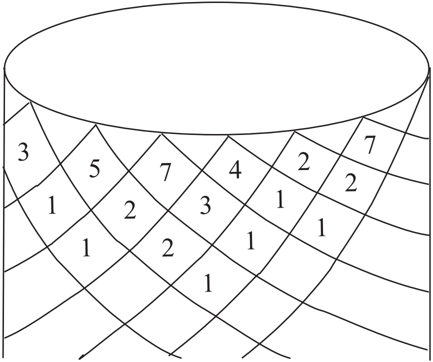


Cylindric partitions and periodic Schur process

[Gessel-Krattenthaler 1997, Borodin 2007]



Picture by Borodin.

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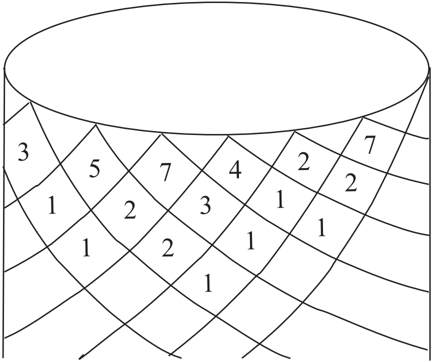

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Periodic Schur process [Borodin 2007]

Consider a periodic sequence $\underline{\lambda} = \lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(2N)} = \lambda^{(0)}$ of integer partitions, and set

$$W(\underline{\lambda}) = s_{\lambda^{(0)}/\lambda^{(1)}}(q) s_{\lambda^{(2)}/\lambda^{(1)}}(q^{-2}) \cdots s_{\lambda^{(2N)}/\lambda^{(2N-1)}}(q^{-2N}) \times q^{2N|\lambda^{(0)}|}$$

Proposition

The weight $W(\underline{\lambda})$ is nonzero iff $\underline{\lambda}$ corresponds to a cylindric partition π, in which case $W(\underline{\lambda}) = q^{\mathrm{vol}(\pi)}$.

- The extra factor $q^{2N|\lambda^{(0)}|}$ is needed, as otherwise constant sequences would all have weight 1.

Different type of boundary conditions

We have encountered instances of Schur process with various types of “boundary conditions”:

- empty/empty [Okounkov-Reshetikhin 2003],
- free/empty [Borodin-Rains 2005],
- periodic [Borodin 2007].

Missing case: free/free (equivalent to periodic with reflection symmetry).

Schur point processes and correlation functions

We consider the point process:

$$\mathfrak{S}(\underline{\lambda}) = \left\{ (i, \lambda_j^{(i)} - j + 1/2), i \in \mathbb{Z}, j \geq 0 \right\}$$

(related to the position of horizontal lozenges in the tiling picture).

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- empty/empty: determinantal [Okounkov-Reshetikhin 2003],
- free/empty: pfaffian [Borodin-Rains 2005],
- periodic: determinantal after a "shift-mixing" [Borodin 2007],
- free/free: pfaffian after mixing [BBNV17].

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Mixing means we have to consider a vertically-shifted process $\mathfrak{S}(\underline{\lambda}) + (0, c)$ with c random. In the free fermion picture, c arises as the charge.

Determinantal and pfaffian point processes

A simple point process ξ in a discrete space X is said:

- determinantal if

$$\mathrm{Prob}(\{x_1, \dots, x_n\} \subset \xi) = \det_{1 \le i,j \le n} k(x_i, x_j)$$

  for some $k : X \times X \to \mathbb{C}$,

- pfaffian if

$$\mathrm{Prob}(\{x_1, \dots, x_n\} \subset \xi) = \mathrm{pf}[K(x_i, x_j)]_{1 \le i,j \le n}$$

  for some $K : X \times X \to M_2(\mathbb{C})$ with $K(x,y) = -K(y,x)^T$,

for any finite set $\{x_1, \dots, x_n\} \subset X$.

Determinantal is a subcase of pfaffian, when taking

$$K(x,y) = \begin{pmatrix} 0 & k(x,y) \\ -k(y,x) & 0 \end{pmatrix}.$$

Outline

General definition

The free boundary Schur process is a random sequence of partitions

$$\mu^{(0)} \subset \lambda^{(1)} \supset \mu^{(1)} \subset \cdots \supset \mu^{(N-1)} \subset \lambda^{(N)} \supset \mu^{(N)}$$

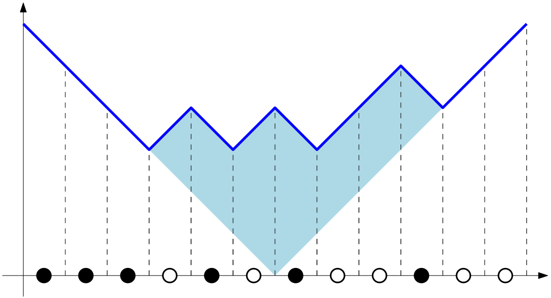
such that

$$\mathrm{Prob}(\underline{\lambda}, \underline{\mu}) = \frac{1}{Z} u^{|\mu^{(0)}|} v^{|\mu^{(N)}|} \prod_{k=1}^{N} \left(s_{\lambda^{(k)}/\mu^{(k-1)}} \left(\rho_k^+\right) s_{\lambda^{(k)}/\mu^{(k)}} \left(\rho_k^-\right) \right).$$

Here:

- u, v are nonnegative real parameters (recover empty boundary conditions by taking them zero),
- the $\rho_k^\pm$ are collections of variables (e.g. single variables for plane partitions),
- $Z = Z(u, v, \ldots)$ is the *partition function*.

Partitions and fermionic states



There is a well-known correspondence between:

- charged partitions (λ, c) with λ a partition and c an integer "charge"
- Maya diagrams, i.e. subsets S of $\mathbb{Z}' := \mathbb{Z} + 1/2$ such that S has a largest element and $\mathbb{Z}' \setminus S$ a smallest element.

This mapping reads explicitly $(\lambda, c) \mapsto \{\lambda_i - i + c + 1/2, i \geq 1\}$ hence is closely related to point configurations.

Fock space treatment

Let ψ_k, ψ_k^* ($k \in \mathbb{Z}'$) be the fermionic operators satisfying the canonical anticommutation relations

$$\psi_k \psi_\ell^* + \psi_\ell^* \psi_k = \delta_{k,\ell}$$

and the vacua $\langle 0|$, $|0\rangle$ such that

$$\langle 0|\psi_k = \langle 0|\psi_{-k}^* = 0, \qquad \psi_k^*|0\rangle = \psi_{-k}|0\rangle = 0, \qquad k > 0.$$

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The action of fermionic operators on $|0\rangle$ generates the Fock space $\mathcal{F}$ whose basis is indexed by Maya diagrams:

$$|S\rangle = |\lambda, c\rangle = (-1)^{j_1 + \cdots + j_r + r/2} \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^* |0\rangle$$

where $i_1 > \cdots > i_r > 0 > j_1 > \cdots > j_s$ and $c = r - s$.

Fock space and Schur functions

The bosonic operators $\alpha_n := \sum_{k \in \mathbb{Z}'} \psi_{k-n} \psi_k^*$ ($n \neq 0$) generate a Heisenberg algebra, and the (half-)vertex operators

$$\Gamma_\pm(\rho) := \exp\left(\sum_{n \geq 1} \frac{p_n(\rho) \alpha_{\pm n}}{n} \right)$$

have skew Schur functions as their matrix elements:

$$\langle \lambda, c | \Gamma_+(\rho) | \mu, c' \rangle = \langle \mu, c' | \Gamma_-(\rho) | \lambda, c \rangle = \begin{cases} s_{\mu/\lambda}(\rho), & \text{if } c = c', \\ 0, & \text{otherwise.} \end{cases}$$

[see e.g. Jimbo-Miwa 1983, Miwa-Jimbo-Date 2000]

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[see e.g. Jimbo-Miwa 1983, Miwa-Jimbo-Date 2000]

The partition function of the Schur process with empty b.c. reads

$$Z = \langle 0 | \Gamma_+(\rho_1^+) \Gamma_-(\rho_1^-) \cdots \Gamma_+(\rho_N^+) \Gamma_-(\rho_N^-) | 0 \rangle.$$

[Okounkov-Reshetikhin 2003]

Free boundary states [B.-Chapuy-Corteel 2014]

To handle free boundaries, we introduce free boundary states

$$|\underline{v}\rangle := \sum_\lambda v^{|\lambda|} |\lambda, 0\rangle, \qquad \langle \underline{u}| := \sum_\lambda u^{|\lambda|} \langle \lambda, 0|$$

and then the partition function reads

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We have the following reflection identities:

$$\Gamma_+(\rho)|\underline{v}\rangle = \tilde{H}(v\rho)\Gamma_-(v^2\rho)|\underline{v}\rangle, \qquad \langle \underline{u}|\Gamma_-(\rho) = \tilde{H}(u\rho)\Gamma_+(u^2\rho)\langle \underline{u}|.$$

where

$$\tilde{H}(\rho) := \sum_\lambda s_\lambda(\rho) = \prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} \qquad \text{(Littlewood identity).}$$

The partition function

Using the reflection identities together with the more standard relations

$$\Gamma_+(\rho)\Gamma_-(\rho') = H(\rho;\rho')\Gamma_-(\rho')\Gamma_+(\rho), \quad \Gamma_+(\rho)|0\rangle = |0\rangle, \quad \langle 0|\Gamma_-(\rho') = \langle 0|$$

where

$$H(\rho;\rho') := \sum_\lambda s_\lambda(\rho)s_\lambda(\rho') = \prod_{i,j}\frac{1}{1-x_iy_j} \qquad \text{(Cauchy identity)}$$

we get

Theorem [B.-Chapuy-Corteel 2014, BBNV17]

$$Z = \prod_{1\leq k\leq \ell\leq N} H(\rho_k^+;\rho_\ell^-)\prod_{n\geq 1}\frac{\tilde{H}(u^{n-1}v^n\rho^+)\tilde{H}(u^nv^{n-1}\rho^-)H(u^{2n}\rho^+;v^{2n}\rho^-)}{1-u^nv^n}$$

where $\rho^\pm = \rho_1^\pm \cup \rho_2^\pm \cup \cdots \cup \rho_N^\pm$.

Particular cases

- The original Schur process $u = v = 0$:

$$Z = \prod_{1 \leq k \leq \ell \leq N} H(\rho_k^+; \rho_\ell^-)$$

Plane partitions:

$$Z = H(q^{1/2}, q^{3/2}, \ldots; q^{1/2}, q^{3/2}, \ldots) = \prod_{j \geq 1} \frac{1}{(1 - q^j)^j}.$$

- The pfaffian Schur process $u = 0$, $v = 1$:

$$Z = \prod_{1 \leq k \leq \ell \leq N} H(\rho_k^+; \rho_\ell^-) \times \tilde{H}(\rho^+)$$

Symmetric plane partitions:

$$Z = \tilde{H}(q, q^3, \ldots) = \prod_{j \geq 1} \frac{1}{(1 - q^{2j-1}) \times (1 - q^{2j})^{2j-2}}.$$

Outline

Correlation functions

The general correlation function is obtained by inserting some "observables" $\psi_k \psi_k^*$ at appropriate places within

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But the naive generalization of Wick's theorem to free boundary states fails. We solve this problem by introducing "extended" free boundary states, which are not eigenvalues of the charge operator.

Extended free boundary states

Let

$$X(v,t) := t \sum_{k>\ell>0} v^{k+\ell}\psi_k\psi_\ell + \sum_{k>0>\ell} (-1)^{\ell+1/2} v^{k-\ell}\psi_k\psi_\ell^* + t^{-1} \sum_{0>k>>\ell} (-1)^{k+\ell+1} v^{-k-\ell}\psi_k^*\psi_\ell^*.$$

Then we have

$$|\underline{v,t}\rangle := e^{X(v,t)}|0\rangle = \sum_\lambda \sum_{c\in 2\mathbb{Z}} t^{c/2} v^{|\lambda|+c^2/2}|\lambda, c\rangle.$$

In particular, $|\underline{v}\rangle$ is the projection of $|\underline{v,t}\rangle$ on the subspace of charge 0. We construct $\langle \underline{u,t}|$ similarly.

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We note that $X(v,t)$ belongs to the Lie algebra D'_∞, this amounts to a fermionic Bogoliubov transformation.

Extended free boundary states

Wick's theorem for free boundaries [BBNV 2017]

Let Ψ be again the vector space spanned by (possibly infinite linear combinations of) the ψ_k and ψ_k^*, $k \in \mathbb{Z}'$. For $\phi_1, \ldots, \phi_{2n} \in \Psi$ and $uv < 1$, we have

$$\frac{\langle \underline{u,t} | \phi_1 \cdots \phi_{2n} | \underline{v,t} \rangle}{\langle \underline{u,t} | \underline{v,t} \rangle} = \mathrm{pf}\, A \qquad (1)$$

where A is the antisymmetric matrix defined by
$A_{ij} = \langle \underline{u,t} | \phi_i \phi_j | \underline{v,t} \rangle / \langle \underline{u,t} | \underline{v,t} \rangle$ for $i < j$.

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Passing to extended free boundary states amounts to randomly moving the point configuration by an even vertical shift c with

$$\mathrm{Prob}(c) = \frac{t^c (uv)^{c^2/2}}{\theta_3(t^2; (uv)^4)}.$$

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Wick's theorem implies that this process is pfaffian (not determinantal since $\langle \psi\psi \rangle \neq 0$). Note that there is no shift for $uv = 0$ (one free boundary).

Extended free boundary states

Some further remarks:

- we have the fermionic reflection relations

$$\psi(z)|\underline{v,t}\rangle = t^{-1}\frac{v-z}{v+z}\psi^*\left(\frac{v^2}{z}\right)|\underline{v,t}\rangle \qquad \begin{cases}\psi(z) := \sum \psi_k z^k \\ \psi^*(w) := \sum \psi_k^* w^{-k}\end{cases}$$

- $|\underline{v,t}\rangle$ is closely related to a sum over states of any charge, which we can view as a pure tensor:

$$\begin{aligned}|\widehat{v,s}\rangle &:= \sum_\lambda \sum_{c\in\mathbb{Z}} s^c v^{|\lambda|+c^2/2}|\lambda,c\rangle \\ &= \prod_{k\in\mathbb{Z}'_-}^{\otimes}\left(s^{-1}v^{-k}|\circ_k\rangle + |\bullet_k\rangle\right)\prod_{k\in\mathbb{Z}'_+}^{\otimes}\left(|\circ_k\rangle + sv^k|\bullet_k\rangle\right).\end{aligned}$$

- fermionic propagators can be evaluated using this representation, or the boson-fermion correspondence (bosonization).

Correlation functions

Recall the definition of the point configuration

$$\mathfrak{S}(\underline{\lambda}) := \left\{ \left(i, \lambda_j^{(i)} - j + \frac{1}{2} \right), 1 \leq i \leq N, j \geq 1 \right\} \subset \mathbb{Z} \times \left(\mathbb{Z} + \frac{1}{2} \right)$$

and denote by $\mathfrak{S}'(\underline{\lambda}) := \mathfrak{S}(\underline{\lambda}) + (0, c)$ the shifted point configuration.

We show that

$$\mathrm{Prob}(\{x_1, \ldots, x_n\} \subset \mathfrak{S}'(\underline{\lambda})) = \mathrm{pf}[K(x_i, x_j)]_{1 \leq i,j \leq n}$$

for some explicit correlation kernel K.

Correlation kernel

It takes the form

$$K_{1,1}(i,k;i',k') = \left[z^k w^{k'}\right] F(i,z)F(i',w)\kappa_{1,1}(z,w)$$

$$K_{1,2}(i,k;i',k') = \left[z^k w^{-k'}\right] \frac{F(i,z)}{F(i',w)}\kappa_{1,2}(z,w)$$

$$K_{2,2}(i,k;i',k') = \left[z^{-k} w^{-k'}\right] \frac{1}{F(i,z)F(i',w)}\kappa_{2,2}(z,w)$$

where:

- F and κ are Laurent series in z and w (obtained as expansions of meromorphic functions in certain compatible annuli)
- only F depends on the $\rho_k^{\pm}$ ("dressing")
- the κ's encodes the boundary conditions: $\kappa_{1,1}(z,w) = \langle \underline{u,t} | \psi(z)\psi(w) | \underline{v,t} \rangle$, etc.

Correlation functions: one free boundary

Theorem [Borodin-Raines 2005, Ghosal 2017, BBNV 2017]

For $u = 0$, the point process $\mathfrak{S}(\underline{\lambda})$ is pfaffian, and its correlation kernel takes the universal form with

$$F(i,z) = \frac{\prod_{1\le\ell\le i} H(\rho_\ell^+; z)}{H(v^2\rho^+; z^{-1}) \prod_{i\le\ell\le N} H(\rho_\ell^-; z^{-1})}$$

$$\kappa_{1,1}(z,w) = \frac{v^2(z-w)\sqrt{zw}}{(z+v)(w+v)(zw-v^2)}$$

$$\kappa_{1,2}(z,w) = \frac{(zw-v^2)\sqrt{zw}}{(z+v)(w-v)(z-w)}$$

$$\kappa_{2,2}(z,w) = \frac{v^2(z-w)\sqrt{zw}}{(z-v)(w-v)(zw-v^2)}.$$

We shall expand the κ's in the annuli $|z|, |w| > v$, with $|z| > |w|$ for $i \le i'$ and vice versa otherwise.

Correlation functions: one free boundary

Remarks:

- in [Borodin-Raines 2005], the expressions appear slightly different because the first partition is assumed to have even columns,
- for $v = 0$, the diagonal entries $K_{1,1}$ and $K_{2,2}$ vanish, and we recover the result from [Okounkov-Reshetikhin 2003] that $\mathfrak{S}(\underline{\lambda})$ is determinantal with kernel

$$k(i,k;i',k') = \left[z^k w^{-k'}\right] \frac{F(i,z)}{F(i',w)} \frac{\sqrt{zw}}{(z-w)}$$

where

$$F(i,z) := \frac{\prod_{1 \le \ell \le i} H(\rho_\ell^+; z)}{\prod_{i \le \ell \le N} H(\rho_\ell^-; z^{-1})}.$$

Correlation function: two free boundaries [BBNV 2017]

We find that $\mathfrak{S}(\underline{\lambda})$ is pfaffian, and its correlation kernel takes the universal form with

$$F(i,z) = \frac{\prod_{1\le \ell \le i} H(\rho_\ell^+;z)}{\prod_{i\le \ell\le N} H(\rho_\ell^-;z^{-1})} \prod_{n\ge 1} \frac{H(u^{2n}v^{2n-2}\rho^-;z)H(u^{2n}v^{2n}\rho^+;z)}{H(u^{2n-2}v^{2n}\rho^+;z^{-1})H(u^{2n}v^{2n}\rho^-;z^{-1})}$$

$$\kappa_{1,1}(z,w) = \frac{v^2}{tz^{1/2}w^{3/2}} \frac{(u^2v^2;u^2v^2)_\infty^2}{(uz,uw,-\frac{v}{z},-\frac{v}{w};uv)_\infty} \frac{\theta_{u^2v^2}(\frac{w}{z})}{\theta_{u^2v^2}(u^2zw)} \frac{\theta_3\left((\frac{tzw}{v^2})^2;u^4v^4\right)}{\theta_3(t^2;u^4v^4)}$$

$$\kappa_{1,2}(z,w) = \frac{w^{1/2}}{z^{1/2}} \frac{(u^2v^2;u^2v^2)_\infty^2}{(uz,-uw,-\frac{v}{z},\frac{v}{w};uv)_\infty} \frac{\theta_{u^2v^2}(u^2zw)}{\theta_{u^2v^2}(\frac{w}{z})} \frac{\theta_3\left((\frac{tz}{w})^2;u^4v^4\right)}{\theta_3(t^2;u^4v^4)}$$

$$\kappa_{2,2}(z,w) = \frac{tv^2}{z^{1/2}w^{3/2}} \frac{(u^2v^2;u^2v^2)_\infty^2}{(-uz,-uw,\frac{v}{z},\frac{v}{w};uv)_\infty} \frac{\theta_{u^2v^2}(\frac{w}{z})}{\theta_{u^2v^2}(u^2zw)} \frac{\theta_3\left((\frac{tv^2}{zw})^2;u^4v^4\right)}{\theta_3(t^2;u^4v^4)}$$

where $(a_1,\dots,a_m;q)_\infty := \prod_{k=0}^\infty (1-a_1q^k)\cdots(1-a_mq^k)$ and $\theta_q(z) := (z;q)_\infty(q/z;q)_\infty$. Laurent expansion is for $u^{-1} > |z|,|w| > v$.

Correlation function: two free boundaries

- For $u = 0$ we recover the previous case of one free boundary.
- It is actually possible to evaluate the pfaffians and express the general n-point correlation for both $\mathfrak{S}(\underline{\lambda})$ and $\mathfrak{S}_d(\underline{\lambda})$ as a coefficient in a Laurent series in $2n$ variables. Behind this, there is an elliptic pfaffian identity which can be rewritten as a particular case of an identify due to Okada (2006). For $u = 0$, we recover Schur's pfaffian identity

$$\operatorname*{pf}_{1\le i<j\le 2n} \frac{x_i - x_j}{x_i + x_j} = \prod_{1\le i<j\le 2n} \frac{x_i - x_j}{x_i + x_j}$$

by a simple change of variables.

An elliptic pfaffian identity

$$\operatorname{pf}\begin{bmatrix} \kappa_{1,1}(z_i,z_j) & \kappa_{1,2}(z_i,w_j) \\ -\kappa_{1,2}(z_j,w_i) & \kappa_{2,2}(w_i,w_j) \end{bmatrix}_{1\le i,j\le n} = \sqrt{\frac{w_1\cdots w_n}{z_1\cdots z_n}} \times$$

$$\frac{((uv)^2;(uv)^2)_\infty^{2n}}{\prod_{i=1}^n (uz_i, -uw_i, -vz_i^{-1}, vw_i^{-1}; uv)_\infty} \frac{\theta_3\left(\left(t\frac{z_1\cdots z_n}{w_1\cdots w_n}\right)^2; (uv)^4\right)}{\theta_3(t^2;(uv)^4)} \times$$

$$\frac{\prod_{i,j=1}^n \theta_{(uv)^2}(u^2 z_i w_j)}{\prod_{1\le i\le j\le n}\theta_{(uv)^2}(w_j/z_i)\prod_{1\le i<j\le n}\theta_{(uv)^2}(z_j/w_i)} \times$$

$$\prod_{1\le i<j\le n} \frac{\theta_{(uv)^2}(z_j/z_i)\theta_{(uv)^2}(w_j/w_i)}{\theta_{(uv)^2}(u^2 z_i z_j)\theta_{(uv)^2}(u^2 w_i w_j)}.$$

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$$\prod_{1\le i<j\le n} \frac{\theta_{(uv)^2}(z_j/z_i)\theta_{(uv)^2}(w_j/w_i)}{\theta_{(uv)^2}(u^2 z_i z_j)\theta_{(uv)^2}(u^2 w_i w_j)}.$$

Conclusion

- We define the free boundary Schur process which generalizes the "original" process of Okounkov-Reshetikhin, and its "pfaffian" variant,
- partition function and correlations are explicitly computed,
- the proof uses the Fock space formalism, with some new tricks
- applications to symmetric plane partitions and also plane overpartitions/domino tilings and last passage percolation
- more work needed to analyze asymptotics in the case with two free boundaries ($uv > 0$). New universality classes?
- are there non free fermionic deformation ? (see e.g. [Barraquand-Borodin-Corwin-Wheeler 2017] for one free boundary)
- Happy birthday Jean-Michel!