Dynamics from Integrability: the Dream of Analytical Solutions

Conference on the occasion of the 60th birthday of Jean Michel Maillet, ENS Lyon, 24 October 2017

Jean-Sébastien Caux
Universiteit van Amsterdam

Work done in collaboration with (among others):

Computation of Dynamical Correlation Functions of Heisenberg Chains in a Magnetic Field

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We compute the momentum-dependent dynamical correlation functions of Heisenberg chains for small wave vectors, taking into account the effect of a transverse magnetic field.

Dynamical structure factor at small $q$ for the XXZ spin-1/2 chain

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Received 29 June 2007.
A long and productive relationship
The bet/challenge

Frank Göhmann to me (Montreal, 2008):

J-S, it’s all very nice what you do,
but it’s not really what we’re looking for.

Me to Jean Michel (about 10 years ago):

At your retirement party, I will remind you that you are still trying to provide an analytical answer to the question of dynamical correlations in quantum spin chains.
Heisenberg spin chain

\[ S(k, \omega), \quad \Delta = 1, \quad h = 0 \]
Gapless XXZ AFM: analytics using vertex operator approach


We consider the XXZ in zero field,

\[
H = J \sum_{j=1}^{N} \left( S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)
\]

\[0 \leq \Delta \leq 1\]

Longitudinal structure factor:

\[
S^{zz}(k, \omega) = \frac{1}{N} \sum_{j,j'} e^{-ik(j-j')} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle S_j^z(t) S_{j'}^z(0) \rangle
\]
Longitudinal structure factor

Separates into

\[ S_{zz}^{zz}(k, \omega) = \sum_{m=1}^{\infty} S_{(2m)}^{zz}(k, \omega) \]

Matrix elements: from vertex operator approach

Jimbo, Miwa, Lashkevich, Pugai, Kojima, Konno, Weston, JSC

\[ S_{2}^{zz}(k, \omega) = \frac{\Theta(\omega_{2,u}(k) - \omega)\Theta(\omega - \omega_{2,l}(k))}{\sqrt{\omega_{2,u}^{2}(k) - \omega^{2}}} \times \left(1 + \frac{1}{\xi}\right)^{2} \frac{e^{-I_{\xi}(\rho(k,\omega))}}{\cosh\frac{2\pi\rho(k,\omega)}{\xi} + \cos\frac{\pi}{\xi}} \]

where

\[ \xi = \frac{\pi}{\text{acos}\Delta} - 1 \quad \cosh(\pi \rho(k,\omega)) = \sqrt{\frac{\omega_{2,u}^{2}(k) - \omega_{2,l}^{2}(k)}{\omega^{2} - \omega_{2,l}^{2}(k)}} \]

\[ I_{\xi}(\rho) \equiv \int_{0}^{\infty} \frac{dt \sinh(\xi + 1)t \cosh(2t)\cos(4\rho t) - 1}{t \sinh \xi t} \cosh t \sinh(2t) \]
Integrability ‘probed’ in the lab

Quantum magnetism

Ultracold atoms

Hope: also Quantum dots, NV centers, Atomic nuclei
Asymptotics: Luttinger Liquids, Correlation Prefactors, Nonlinear LL
Luttinger liquid phenomenology
(Haldane 1981)

Luttinger liquid Hamiltonian:

\[ H_0 = \frac{v}{2\pi} \int dx \left( K (\nabla \theta)^2 + \frac{1}{K} (\nabla \phi)^2 \right) \]

\[ [\phi(x), \nabla \theta(x')] = i\pi \delta(x - x') \]

Haldane’s great insight: **generic in 1d**.
Sound velocity and Luttinger parameter fixed from observables.

*all correlation functions at low energies*
Luttinger liquids: correlators

Around momentum \((2m + 1/2 \pm 1/2) k_F\) the fields are represented as

\[
\psi_{F(B)}(x, t) \sim e^{i(2m+1/2\pm1/2)[k_F x - \phi(x,t)] + i\theta(x,t)}
\]

LL theory predicts the asymptotics \(\rho_0 x \gg 1\) of correlation functions as

\[
\langle \hat{\psi}^\dagger_F(x) \hat{\psi}_F(0) \rangle \approx \sum_{m \geq 0} \frac{K}{2(\pi \rho_0 x)^2} \frac{A_m \cos(2m k_F x)}{(\rho_0 x)^{2m^2 K}} \left( \frac{1}{2} \pm \frac{1}{2} \right)^{m+1/2}
\]

\[
\approx \sum_{m \geq 0} \frac{B_m \cos(2m k_F x)}{(\rho_0 x)^{2m^2 K+1/(2K)}}
\]

\[
\approx \sum_{m \geq 0} \frac{C_m \sin [(2m + 1) k_F x]}{(\rho_0 x)^{(2m+1)^2 K/2+1/(2K)}}
\]
Asymptotes of static function: determined by correlation around Umklapp modes

\[ S^{\rho\rho}(k, \omega) \]

\[ S^{\psi\psi^\dagger}(k, \omega) \]

\[ -\frac{K}{2\pi^2} \quad A_1 \quad A_2 \]

\[ B_0 \quad B_1 \quad B_2 \]
For the XXZ chain, bosonization similarly gives

\[ S^z(x, t) \sim s_z - \frac{\nabla \phi}{\pi} + e^{i2m[(s_z+1/2)\pi x - \phi(x, t)]} \]

\[ S^+(x, t) \sim e^{-i2m[(s_z+1/2)x - \phi(x, t)] - i\theta(x, t)} \]

with long-distance behaviour of static functions

\[
\langle S^z(x)S^z(0) \rangle = s_z^2 - \frac{K}{2(\pi x)^2} + \sum_{m \geq 1} \frac{D_m \cos(2m(s_z + 1/2)\pi x)}{x^{2m^2 K}} \\
\langle S^+(x)S^-(0) \rangle = (-1)^x \sum_{m \geq 0} \frac{E_m \cos(2m(s_z + 1/2)\pi x)}{x^{2m^2 K + 1/(2K)}}
\]

given in terms of non-universal prefactors D and E
Asymptotes of static function: determined by correlation around Umklapp/pi modes

\[ S_{zzz}(k, \omega) \quad S^{+-}(k, \omega) \]

\[ -\frac{K}{2\pi^2} \quad D_0 \quad E_0 \quad E_1 \]
Nonuniversal prefactors in the correlation functions of one-dimensional quantum liquids

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We develop a general approach to calculating “nonuniversal” prefactors in static and dynamic correlation functions of one-dimensional (1D) quantum liquids at zero temperature by relating them to the finite-size scaling of certain matrix elements (form factors). This represents a powerful tool for extracting data valid in the thermodynamic limit from finite-size effects. As the main application, we consider weakly interacting spinless fermions with an arbitrary pair interaction potential, for which we perturbatively calculate certain prefactors in static and dynamic correlation functions. We also evaluate prefactors of the long-distance behavior of correlation functions nonperturbatively for the exactly solvable Lieb-Liniger model of 1D bosons.

Exact prefactors in static and dynamic correlation functions of one-dimensional quantum integrable models: Applications to the Calogero-Sutherland, Lieb-Liniger, and XXZ models

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In this paper, we demonstrate a recently developed technique which addresses the problem of obtaining nonuniversal prefactors of the correlation functions of one-dimensional (1D) systems at zero temperature. Our approach combines the effective field theory description of generic 1D quantum liquids with the finite-size scaling of form factors (matrix elements) which are obtained using microscopic techniques developed in the context of integrable models. We thus establish exact analytic forms for the prefactors of the long-distance behavior of equal-time correlation functions as well as prefactors of singularities of dynamic response functions. In this paper, our focus is on three specific integrable models: the Calogero-Sutherland, Lieb-Liniger, and XXZ models.
For XXZ (transverse correlations):

Fits with DMRG results of Hikihara & Furusaki
OK, with Luttinger liquid theory, we can describe correlations...

... but what about around here?

... around here?

(Where response is large)
Dynamical correlators: Nonlinear Luttinger Liquid Theory

Glazman, Imambekov, Khodas, Kamenev, Cheianov, Pustilnik, Affleck, Pereira, Sirker, JSC, ...

Observation: acting on ground state, an operator creates (few) high-energy + (many) low-energy excitations

Three subband model

‘Impurity’ branch

Luttinger liquid
Singularity structure of response functions

(Khodas, Pustilnik, Kamenev, Glazman; Imambekov & Glazman)

Dynamical structure factor for interacting bosons

\[ S \sim \frac{1}{|\omega - \epsilon_I|^{\mu_1}} \left( \theta(\epsilon_I - \omega) + \nu_1 \theta(\omega - \epsilon_I) \right) \]

\[ S \sim (\omega - \epsilon_{II})^{\mu_2} \]

Singularity at upper 2p threshold

High-energy tail

Lower threshold
Adilet Imambekov
1982-2012
Now: correlations, everywhere they matter?

Fact: at incommensurate fields, the correlation is mathematically nonvanishing everywhere.

Conjecture: it is also non-smooth everywhere (there are infinitely many intercrossing thresholds).

\[ \forall (k, \omega), \exists \ n < \infty \ | \ \partial^n_{k, \omega} S(k, \omega) \text{ does not exist} \]
Dynamics
Far from the
Ground State
Why be content with ground states?
Open up the sea!

left Fermi sea

right Fermi sea

‘Moses state’
Moses state in XXZ

\[ S^{\pm}(k, \omega) \]

as a function of splitting

Fits with nonlinear Luttinger liquid:

Opportunities for rich analytics here!
Integrable Models Beyond Equilibrium
Out-of-equilibrium using integrability

**Pulsed:**
- The super Tonks-Girardeau gas
- Split Fermi seas (Moses states)
- Spin echo in quantum dots
- Quasisolitons

**Quenched:**
- Interaction quench in Richardson
- Domain wall release in Heisenberg
- Geometric quench
- Interaction cutoff in Lieb-Liniger
- Release of trapped Lieb-Liniger
- BEC to Lieb-Liniger quench
- Quantum Newton’s Cradle in TG
- Néel to XXZ quench
- Generalized hydrodynamics

**Driven:**
- Floquet driving central spin
- Floquet driving Heisenberg
Progress on quenches

The most ‘physical’ GGE

Ilievski, Quinn, Caux PRB 85, 115128 (2017)

The natural basis for all conservation laws: the densities of fundamental particles, as encoded in the distribution of Bethe roots

\[ \hat{\rho}_{GGE} = \frac{1}{Z} \exp \left[ -\sum_j \int_{\mathbb{R}} d\lambda \, \mu_j(\lambda) \hat{\rho}_j(\lambda) \right] \]

This is like a ‘momentum distribution function’ for true particles, which connects smoothly to the noninteracting limit

\[ \Omega_s^{\Psi_0}(\lambda) = \lim_{\text{th}} \frac{\langle \Psi_0 | \hat{X}_s(\lambda) | \Psi_0 \rangle}{N} \]

\( s = \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots \)
Quasisoliton dynamics in spin chains
Solitons (classical)

John Scott Russell: solitary wave of translation (1834)

(Herriot-Watt University)
Solitons (classical)

(Boussinesq)
Korteweg-de Vries equation

$$\partial_t u + u \partial_x u + \delta^2 \partial_x^3 u = 0$$

First simulations:
Fermi-Pasta-Ulam-(Tsingou)
absence of ergodicity
Further simulations:
Zabusky & Kruskal 1965

concept of a soliton

Classical inverse scattering
Quasisoliton scattering (quantum)

Vlijm, Ganahl, Fioretto, Brockmann, Haque, Evertz and Caux, 2015

‘Worldlines’ of colliding wavepackets:

\[ \Delta = 2 \]

Displacement as a function of anisotropy (fixed incoming momenta)

Measured av. Linear fits

Single magnons Bound magnons

\[ -\chi^{(1,1)} \] \[ -\chi^{(2,2)} \]
Spinon dynamics in real space/time

$|\Psi(0)\rangle = \sqrt{2} S_{j_0}^+ |\text{GS}\rangle$

$\langle S_j^z S_0^z \rangle = \frac{1}{2} \langle S_j^z \rangle$

$\approx D_1 \frac{(-1)^j}{j}$

2-spinons: 96.85%
4-spinons: 2.15%

Saturation: 99.00%
In memoriam

Ludvig Dmitrievich Faddeev
23/3/1934 - 26/2/2017
Generalized hydrodynamics
Generalized hydrodynamics (GHD)

O.A. Castro-Alvaredo, B. Doyon and T. Yoshimura, PRX 6, 041065 (2016)
B. Doyon and T. Yoshimura, SciPost Phys. 2, 0

Quench from spatially inhomogeneous state

After initial dephasings: ‘hydrodynamic’ evolution described by local GGE

Two-equation summary:

Local continuity equation (in terms of Bethe root densities)

Local effective velocity
GHD as ‘molecular dynamics’: the flea gas
B. Doyon, T. Yoshimura and JSC, arXiv:1704.05482

- Encode initial state as a gas of quasisolitons
- Loop: evolve, collide and scatter
  (as if quasisolitons were classical particles, using displacement calculated from quantum phase shifts)

As simple as it gets in the integrability business!
Ergodicity (or lack thereof) in interacting quantum systems close to an integrable model

David Weiss’s quantum Newton’s cradle experiment
The flea gas in a force field: simulating the quantum Newton’s cradle

JSC, B. Doyon, J Dubail, R. Konik and T.Yoshimura, to appear

‘Oscillation’-like dynamics at short time scales
The flea gas in a force field: simulating the quantum Newton’s cradle

JSC, B. Doyon, J Dubail, R. Konik, and T. Yoshimura, to appear

‘Relaxation’-like dynamics at long time scales
Floquet dynamics
The simple pendulum on its head

Kapitza pendulum, 1951

Pyotr L. Kapitza
(8/7/1894-8/4/1984)
The Kapitza pendulum
Floquet basics

Unitary time evolution operator under action of periodic Hamiltonian $\hat{H}(t) = \hat{H}(t + T)$

$$\hat{U}(t) = T(e^{-i \int_0^t dt' H(t')}) = \hat{P}(t)e^{-i\hat{H}_F t}$$

"fast motion" operator
$$\hat{P}(t + T) = \hat{P}(t)$$
$$\hat{P}(nT) = 1 \ (n \in \mathbb{Z})$$

Stroboscopic Floquet operator
$$\hat{U}_F \equiv \hat{U}(T) = e^{-i\hat{H}_F T}$$

Diag'n:
$$\hat{H}_F = \sum_n \epsilon_n |\phi_n\rangle\langle\phi_n|$$
$$\hat{U}_F = \sum_n e^{-i\theta_n} |\phi_n\rangle\langle\phi_n|$$

t evolution:
$$|\psi_n(t)\rangle = e^{-i\epsilon_n t} |\phi_n(t)\rangle, \quad |\phi_n(t)\rangle = \hat{P}(t)|\phi_n\rangle$$
“Quench-dequench” Floquet protocol

Let’s consider the simple case of periodically “switching” between two Hamiltonians:

\[ \hat{H}(t) = \begin{cases} \hat{H}_1 & \text{for } 0 < t < \eta T, \\ \hat{H}_2 & \text{for } \eta T < t < T, \end{cases} \]

so

\[ \hat{U}_F \equiv e^{-i\hat{H}_F T} = e^{-i(1-\eta)T\hat{H}_2} e^{-i\eta T \hat{H}_1} \]

One then finds the nice identities

\[ \epsilon_n = \frac{\theta_n}{T} = \langle \phi_n | \hat{H}_F | \phi_n \rangle \quad \frac{\partial \theta_n}{\partial T} = \langle \phi_n | \hat{H}_{avg} | \phi_n \rangle \]

with t-avg Hamiltonian

\[ \hat{H}_{Avg} = \eta \hat{H}_1 + (1 - \eta) \hat{H}_2 \]
Floquet’ing integrable models
P. Claeys and JSC, arXiv:1708.07324

Idea: take a one-parameter family of integrable models
do quench/dequench sequences on this manifold

\[ H_{avg} = \eta H_1 + (1 - \eta) H_2 \]

Integrable manifold

Time-avg Hamiltonian
also integrable
Hamiltonian: as you perhaps guessed, XXZ:
\[ \hat{H}(t) = -J \sum_i \left[ S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta(t) S_i^z S_{i+1}^z \right] \]

Floquet protocol: binary switch between \( \Delta_1 \) and \( \Delta_2 \)

Stroboscopic Floquet operator: \( \hat{U}_F = e^{-i(1-\eta)T\hat{H}_2} e^{-i\eta T\hat{H}_1} \)

No analytical solution, must rely on numerics

Focus on sector with \( k = 0 \) and \( m_z = 1/3 \)
Floquet phases, quasienergies and avg quasienergies as a function of $T$

All crossings are avoided
Eigenvalue statistics

\[ r = \frac{\min(s_n, s_{n+1})}{\max(s_n, s_{n+1})} \in [0, 1], \quad s_n = E_{n+1} - E_n \]

Expect
\[ \langle r \rangle_{GOE} \approx 0.535989 \]
\[ \langle r \rangle_{POI} \approx 0.386295 \]

Quasienergies mark transition from Poisson (high frequency) to GOE (low freq)

\[ \langle r(\theta_n/T) \rangle \text{ (symbols)} \]
\[ \langle r(\partial_T \theta_n) \rangle \text{ (dashed lines)} \]
Integrability breaking in Richardson-Gaudin
A variational method for integrability breaking in Richardson-Gaudin models


States

\[
|\psi_{RG}\rangle = \prod_{\alpha=1}^{N} \left( \sum_{i=1}^{L} \frac{S_i^\dagger}{\epsilon_i - \lambda_\alpha} \right) |\downarrow \ldots \downarrow\rangle
\]

diagonalize RG systems

\[
\hat{H} = \sum_{i=1}^{L} \eta_i R_i, \quad \eta_i \in \mathbb{R}
\]

with

\[
R_i = S_i^0 + g \sum_{j \neq i}^{L} \frac{1}{\epsilon_i - \epsilon_j} \left[ \frac{1}{2} \left( S_i^\dagger S_j + S_i S_j^\dagger \right) + S_i^0 S_j^0 \right]
\]

provided

\[
1 + \frac{g}{2} \sum_{i=1}^{L} \frac{1}{\epsilon_i - \lambda_\alpha} - g \sum_{\beta \neq \alpha}^{N} \frac{1}{\lambda_\beta - \lambda_\alpha} = 0, \quad \alpha = 1 \ldots N
\]
A variational method for integrability breaking in Richardson-Gaudin models


Central Spin model:

\[ \hat{H}_{cs} = BS_{1}^{z} + g \sum_{i \neq 1}^{L} \frac{\mathbf{S}_{1} \cdot \mathbf{S}_{i}}{\epsilon_{0,1} - \epsilon_{0,i}} \]

Perturbations: single spin, or two-spin terms

\[ \hat{H} = \hat{H}_{cs} + \mu S_{i}^{0}, \quad \hat{H} = \hat{H}_{cs} + \mu \mathbf{S}_{i} \cdot \mathbf{S}_{j} \]

Strategy: define a variational wavefunction using auxiliary quasi-energies as variational parameters, and optimize

\[ E [\psi_{RG}] = \frac{\langle \psi_{RG} | \hat{H} | \psi_{RG} \rangle}{\langle \psi_{RG} | \psi_{RG} \rangle} \]
Parameters of optimized states

Single spin perturbation:

Two-spin perturbation:
Energies of optimized states

Single spin perturbation:

Two-spin perturbation:
Challenges for Jean Michel

- Compute XXZ gapless AFM dynamical correlations at finite field, beyond asymptotics
- Compute dynamical correlations on states other than ground states or thermal states (start: low-entropy ones like Moses states)
- Find exact overlaps of eigenstates of (pairs of) distinct Hamiltonians (and find other exact quench steady states using the Quench Action)
- Find integrable Floquet Hamiltonians
Conclusions

- Equilibrium dynamics in integrable systems
  - Very healthy recent history

- Quench dynamics in integrable systems
  - Surprisingly healthy recent history!
  - Quench Action (no details presented here, look at papers!)
  - Generalized hydrodynamics: from quasisolitons to the flea gas

- Floquet dynamics in integrable systems
  - Prototypical example: periodic quench/dequench in XXZ

- Integrability breaking in Richardson-Gaudin
  - Bethe states as variational wavefunctions
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