Dynamics from Integrability: the Dream of Analytical Solutions

Conference on the occasion of the 60th birthday of Jean Michel Maillet, ENS Lyon, 24 October 2017

Jean-Sébastien Caux
Universiteit van Amsterdam

Nederlandse Organisatie voor Wetenschappelijk Onderzoek

Work done in collaboration with (among others):

A'dam gang: S. E. Tapias Arze, Pieter Claeys, E. Quinn, E. Ilievski
B. Doyon, J. Dubail, H. G. Evertz, M. Haque, R. Konik, T. Yoshimura…

PRL 95, 077201 (2005) PHYSICAL REVIEW LETTERS week ending 12 AUGUST 2005

Computation of Dynamical Correlation Functions of Heisenberg Chains in a Magnetic Field

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We compute the momentum-

Journal of Statistical Mechanics:
An IOP and SISSA journal

Computation of dynami
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PRL 96, 2572

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Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Dynamical structure factor at small q for the XXZ spin-1/2 chain

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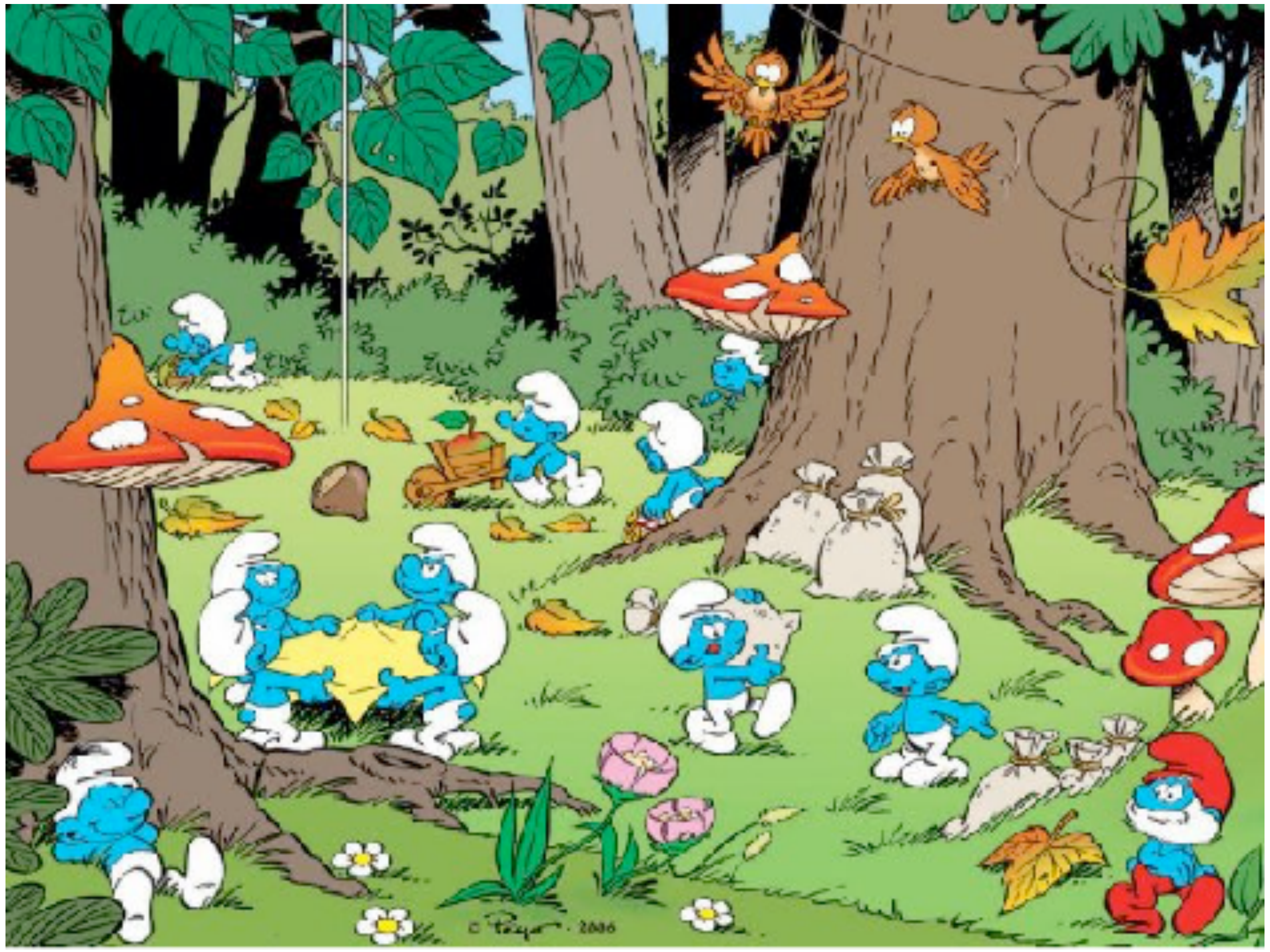
Received 29 June 2007

A long and productive relationship



The bet/challenge

Frank Göhmann to me (Montreal, 2008):

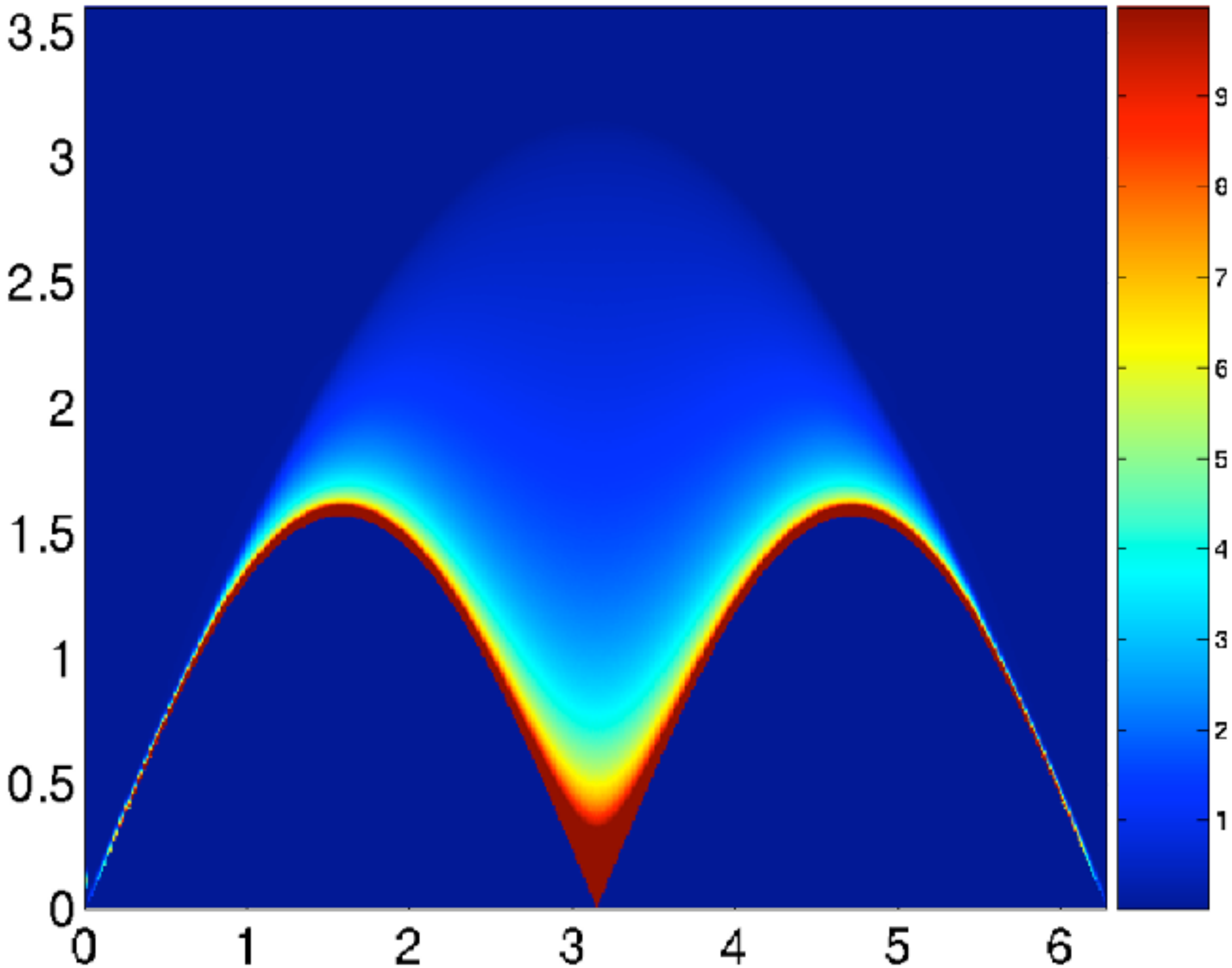
J-S, it's all very nice what you do,
but it's not really what we're looking for.

Me to Jean Michel (about 10 years ago):

At your retirement party, I will remind you that you are still trying to provide an analytical answer to the question of dynamical correlations in quantum spin chains.

Heisenberg spin chain

$S(k, \omega), \quad \Delta = 1, \quad h = 0$



Gapless XXZ AFM: analytics using vertex operator approach

JSC, H. Konno, M. Sorrell and R. Weston, PRL 106, 217203 (2011), JSTAT (2012)

We consider the XXZ in zero field,

$$H = J \sum_{j=1}^{N} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right) \qquad 0 \le \Delta \le 1$$

Longitudinal structure factor:

$$S^{zz}(k,\omega) = \frac{1}{N} \sum_{j,j'} e^{-ik(j-j')} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle S_j^z(t) S_{j'}^z(0) \rangle$$

Longitudinal structure factor

Separates into $$S^{zz}(k,\omega) = \sum_{m=1}^{\infty} S^{zz}_{(2m)}(k,\omega)$$

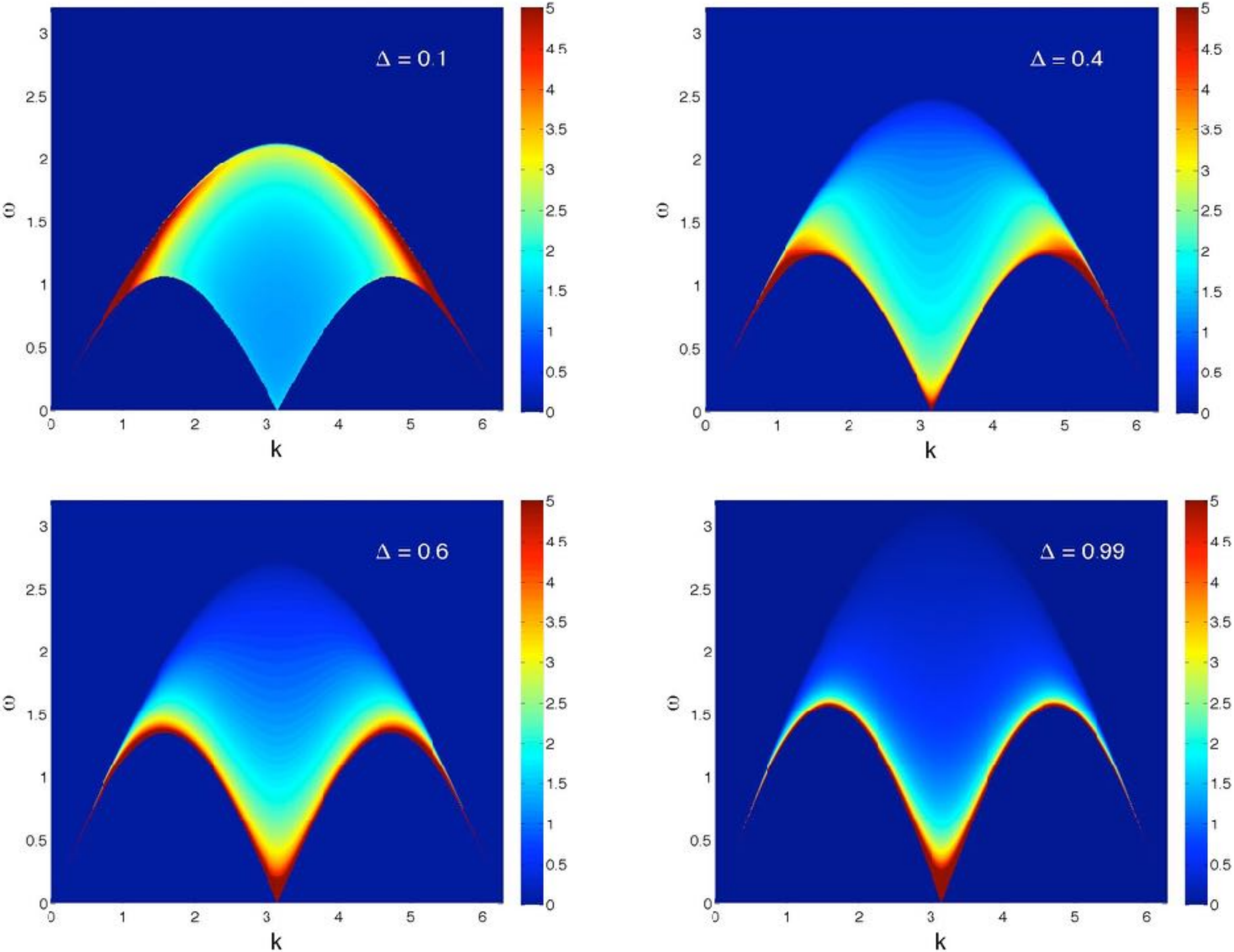
Matrix elements: from vertex operator approach

Jimbo, Miwa, Lashkevich, Pugai, Kojima, Konno, Weston, JSC

$$S_2^{zz}(k,\omega) = \frac{\Theta(\omega_{2,u}(k)-\omega)\Theta(\omega-\omega_{2,l}(k))}{\sqrt{\omega_{2,u}^2(k)-\omega^2}} \times (1+1/\xi)^2 \frac{e^{-I_\xi(\rho(k,\omega))}}{\cosh\frac{2\pi\rho(k,\omega)}{\xi}+\cos\frac{\pi}{\xi}}$$

where $$\xi = \frac{\pi}{\mathrm{acos}\Delta} - 1 \qquad \cosh(\pi\rho(k,\omega)) = \sqrt{\frac{\omega_{2,u}^2(k)-\omega_{2,l}^2(k)}{\omega^2-\omega_{2,l}^2(k)}}$$

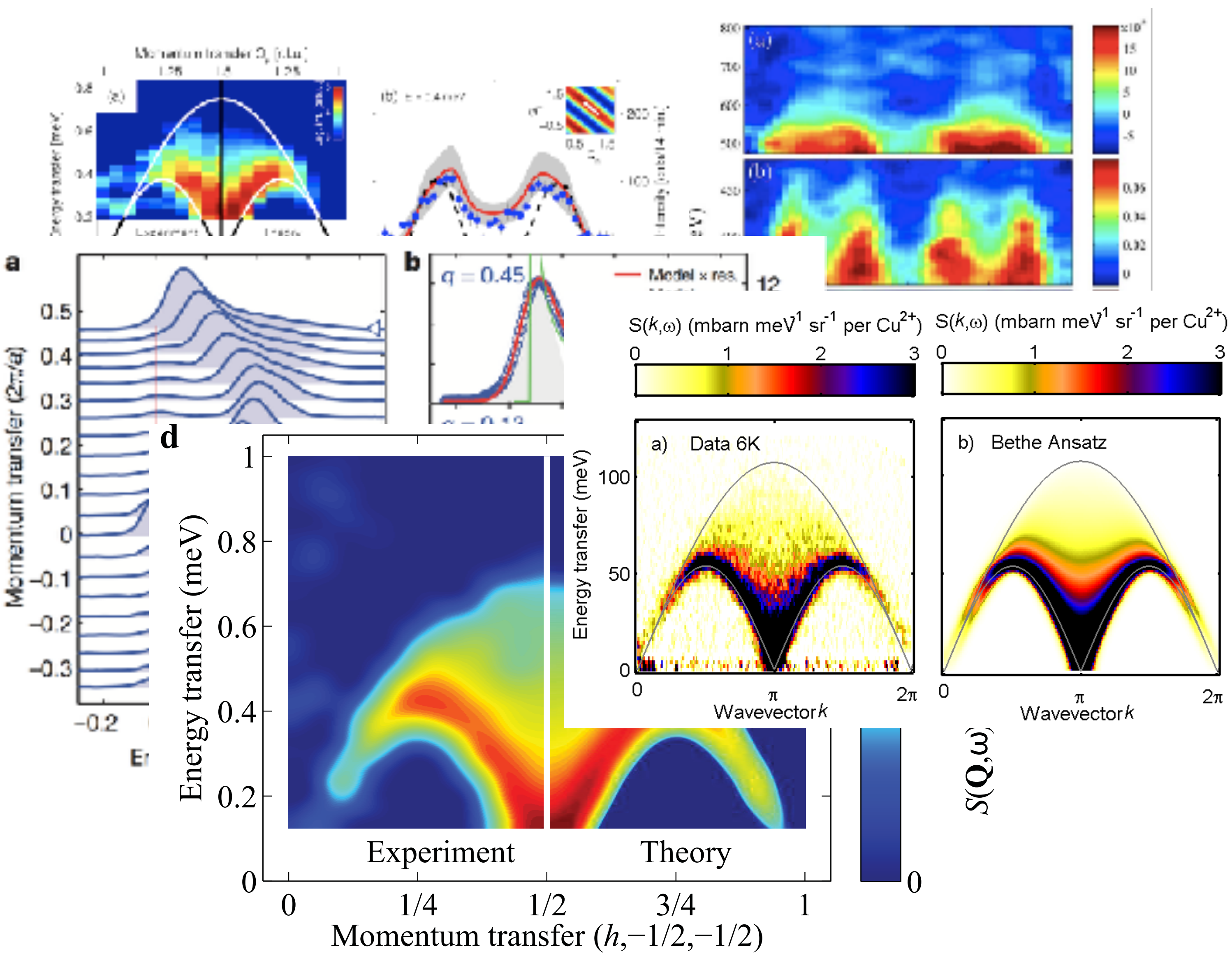
$$I_\xi(\rho) \equiv \int_0^\infty \frac{dt}{t} \frac{\sinh(\xi+1)t}{\sinh \xi t} \frac{\cosh(2t)\cos(4\rho t)-1}{\cosh t \sinh(2t)}$$



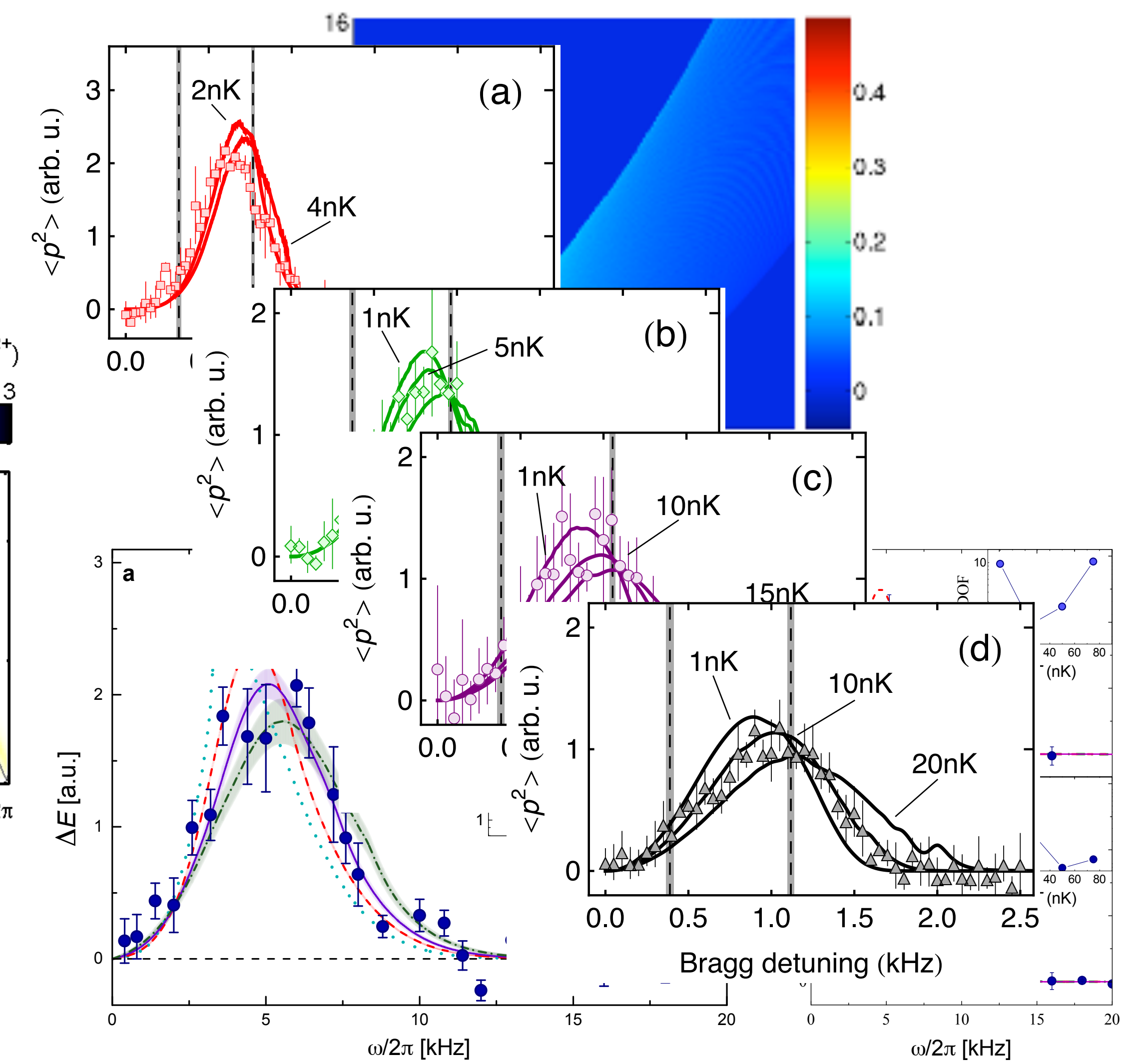
Δ = 0.1
Δ = 0.4
Δ = 0.6
Δ = 0.99
ω
k

Integrability ‘probed’ in the lab

Quantum magnetism



Ultracold atoms



Hope: also Quantum dots, NV centers, Atomic nuclei

Asymptotics: Luttinger Liquids, Correlation Prefactors, Nonlinear LL

Luttinger liquid phenomenology
(Haldane 1981)

Luttinger liquid Hamiltonian:

$$H_0 = \frac{v}{2\pi} \int dx \, \left(K (\nabla \theta)^2 + \frac{1}{K} (\nabla \phi)^2 \right)$$

$$[\phi(x), \nabla\theta(x')] = i\pi\delta(x - x')$$

Haldane's great insight: **generic in 1d**.

Sound velocity and Luttinger parameter fixed from observables.

⇨ ***all correlation functions at low energies***

Luttinger liquids: correlators

Around momentum $(2m + 1/2 \pm 1/2)k_F$

the fields are represented as

$$\psi_{F(B)}(x,t) \sim e^{i(2m+1/2\pm 1/2)[k_F x - \phi(x,t)] + i\theta(x,t)}$$

LL theory predicts the asymptotics $\rho_0 x \gg 1$

of correlation functions as

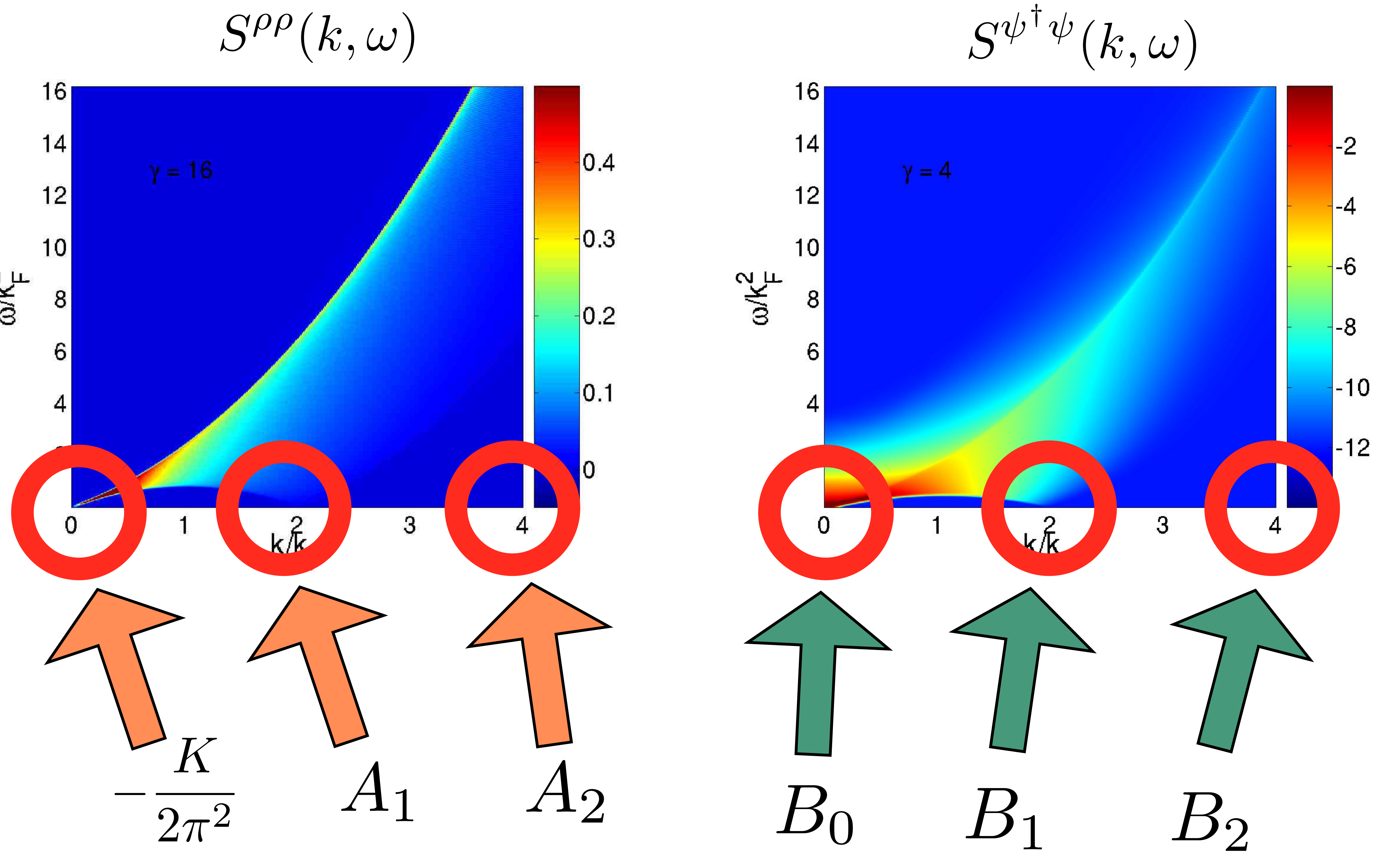
Infinite sets of nonuniversal prefactors

$$\ldots \frac{K}{2(\pi\rho_0 x)^2} + \sum_{m\geq 1} \frac{A_m \cos(2mk_F x)}{(\rho_0 x)^{2m^2 K}}$$

$$\frac{\ldots}{\rho_0} \approx \sum_{m\geq 0} \frac{B_m \cos(2mk_F x)}{(\rho_0 x)^{2m^2 K + 1/(2K)}}$$

$$\frac{\langle \hat{\psi}_F^\dagger(x)\hat{\psi}_F(0)\rangle}{\rho_0} \approx \sum_{m\geq 0} \frac{C_m \sin[(2m+1)k_F x]}{(\rho_0 x)^{(2m+1)^2 K/2 + 1/(2K)}}$$

Asymptotes of static function: determined by correlation around Umklapp modes



For the XXZ chain, bosonization similarly gives

$$S^z(x,t) \sim s_z - \frac{\nabla\phi}{\pi} + e^{i2m[(s_z+1/2)\pi x - \phi(x,t)]}$$

$$S^+(x,t) \sim e^{-i2m[(s_z+1/2)x - \phi(x,t)] - i\theta(x,t)}$$

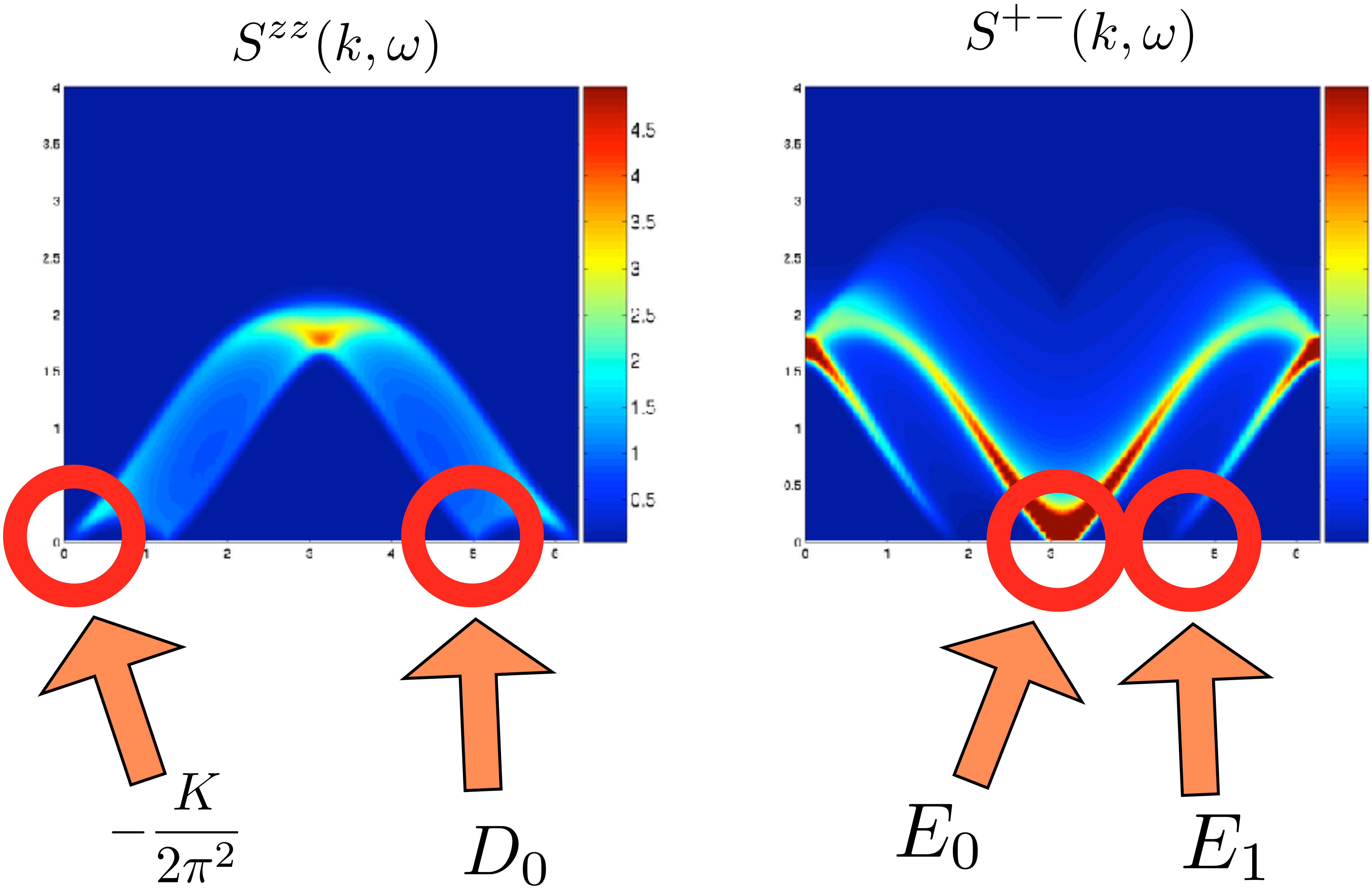
with long-distance behaviour of static functions

$$\langle S^z(x)S^z(0)\rangle = s_z^2 - \frac{K}{2(\pi x)^2} + \sum_{m\geq 1} \frac{D_m cos(2m(s_z+1/2)\pi x)}{x^{2m^2K}}$$

$$\langle S^+(x)S^-(0)\rangle = (-1)^x \sum_{m\geq 0} \frac{E_m cos(2m(s_z+1/2)\pi x)}{x^{2m^2K+1/(2K)}}$$

given in terms of non-universal prefactors D and E

Asymptotes of static function: determined by correlation around Umklapp/pi modes



PHYSICAL REVIEW B **84**, 045408 (2011)

Nonuniversal prefactors in the correlation functions of one-dimensional quantum liquids

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(Received 17 October 2010; revised manuscript received 5 April 2011; published 5 July 2011)

We develop a general approach to calculating "nonuniversal" prefactors in static and dynamic correlation functions of one-dimensional (1D) quantum liquids at zero temperature by relating them to the finite-size scaling of certain matrix elements (form factors). This represents a powerful tool for extracting data valid in the thermodynamic limit from finite-size effects. As the main application, we consider weakly interacting spinless fermions with an arbitrary pair interaction potential, for which we perturbatively calculate certain prefactors in static and dynamic correlation functions. We also evaluate prefactors of the long-distance behavior of correlation functions nonperturbatively for the exactly solvable Lieb-Liniger model of 1D bosons.

PHYSICAL REVIEW B **85**, 155136 (2012)

Exact prefactors in static and dynamic correlation functions of one-dimensional quantum integrable models: Applications to the Calogero-Sutherland, Lieb-Liniger, and XXZ models

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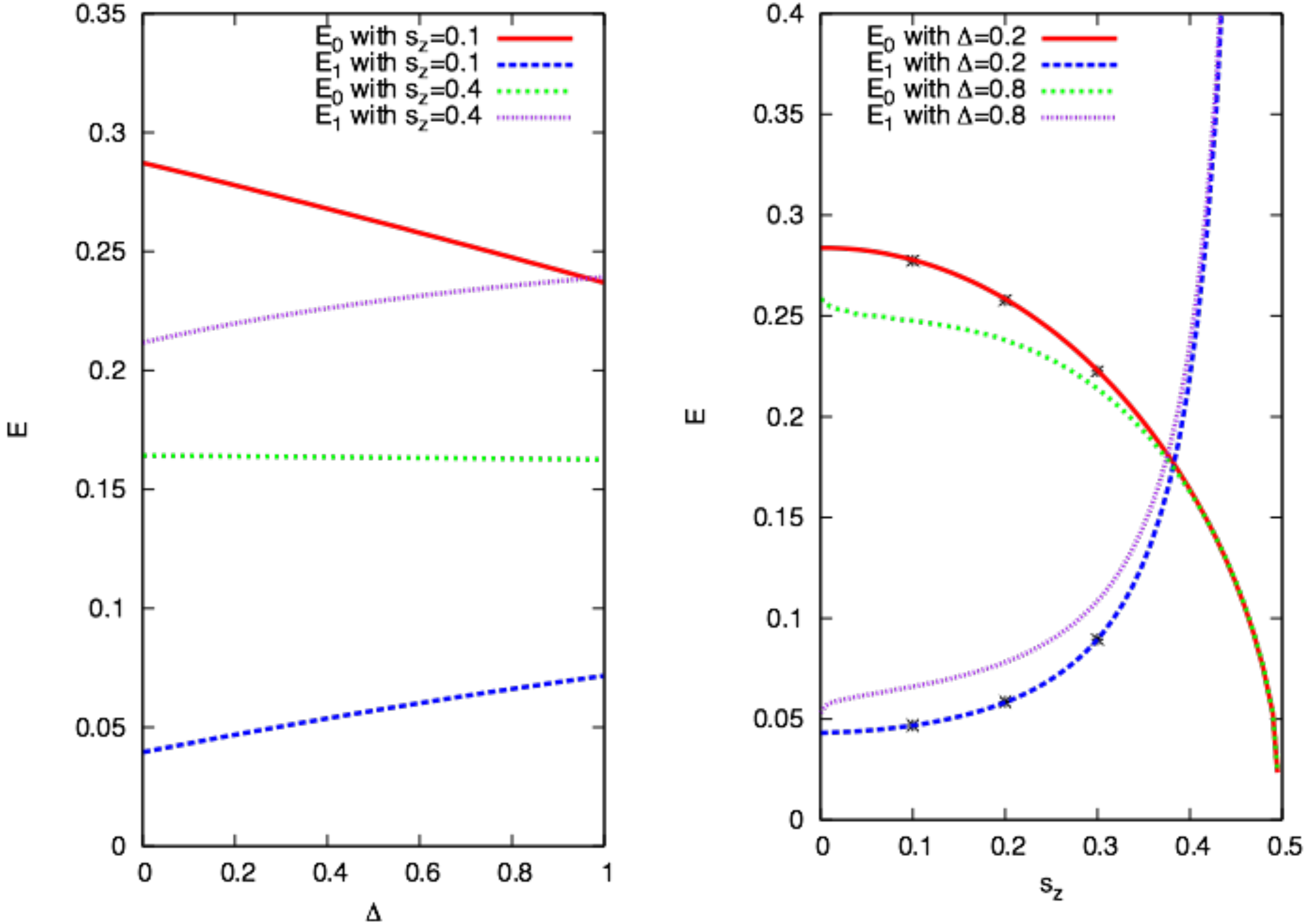
(Received 4 November 2011; revised manuscript received 9 February 2012; published 24 April 2012)

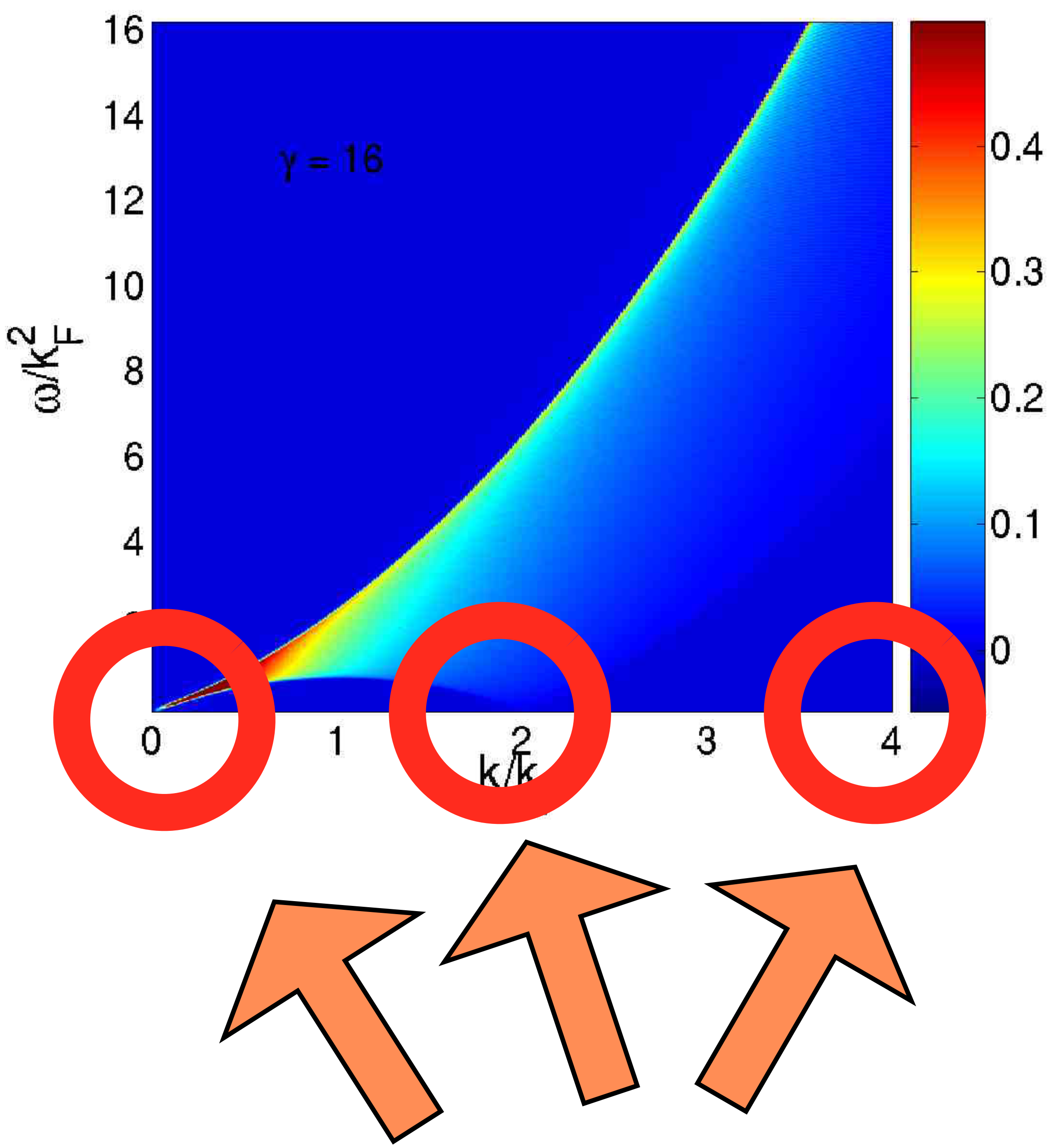
In this paper, we demonstrate a recently developed technique which addresses the problem of obtaining nonuniversal prefactors of the correlation functions of one-dimensional (1D) systems at zero temperature. Our approach combines the effective field theory description of generic 1D quantum liquids with the finite-size scaling of form factors (matrix elements) which are obtained using microscopic techniques developed in the context of integrable models. We thus establish exact analytic forms for the prefactors of the long-distance behavior of equal-time correlation functions as well as prefactors of singularities of dynamic response functions. In this paper, our focus is on three specific integrable models: the Calogero-Sutherland, Lieb-Liniger, and XXZ models.

For XXZ (transverse correlations):



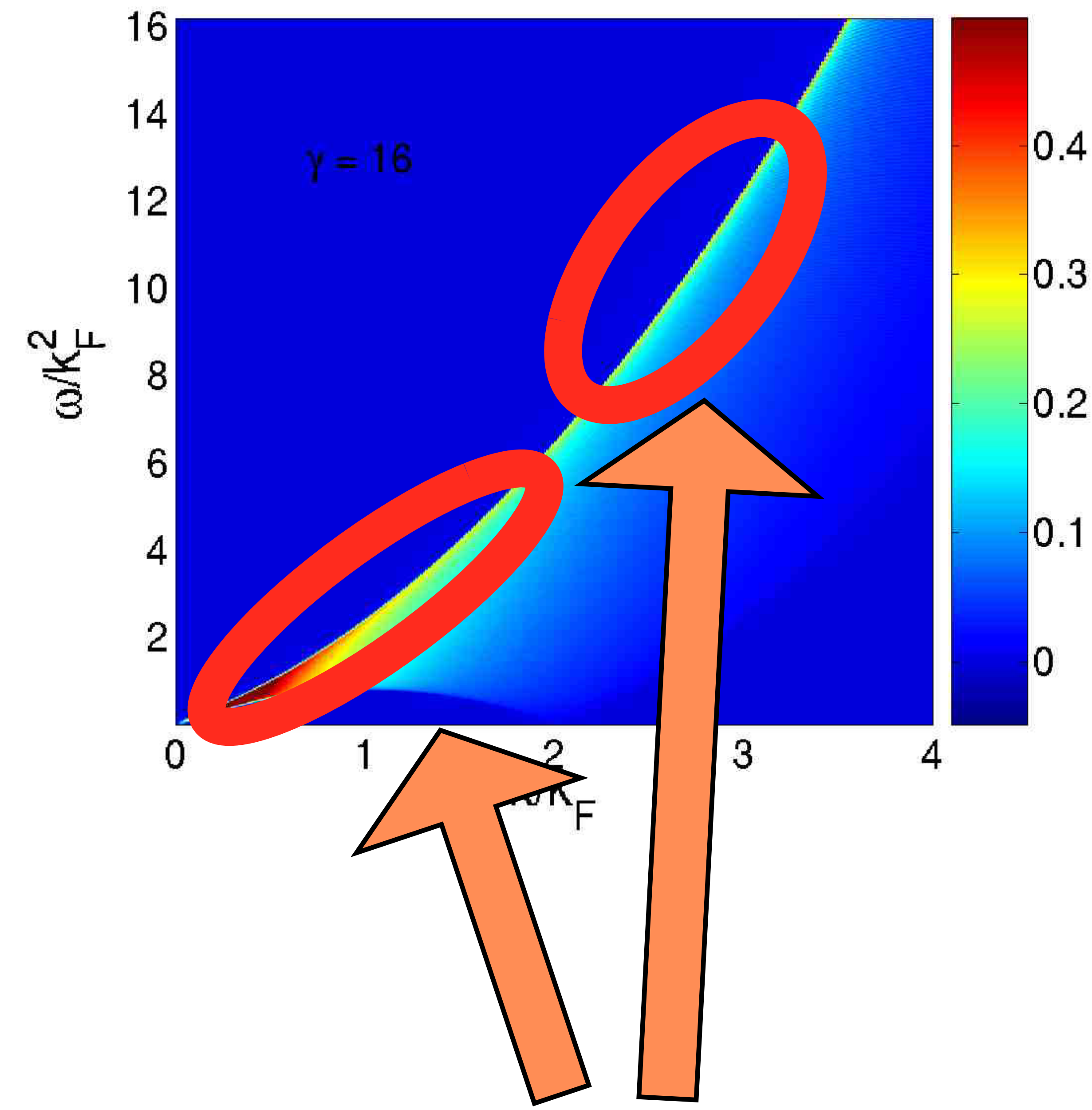
Fits with DMRG results of Hikihara & Furusaki

OK, with Luttinger liquid theory, we can describe correlations...



... around here

... but what about



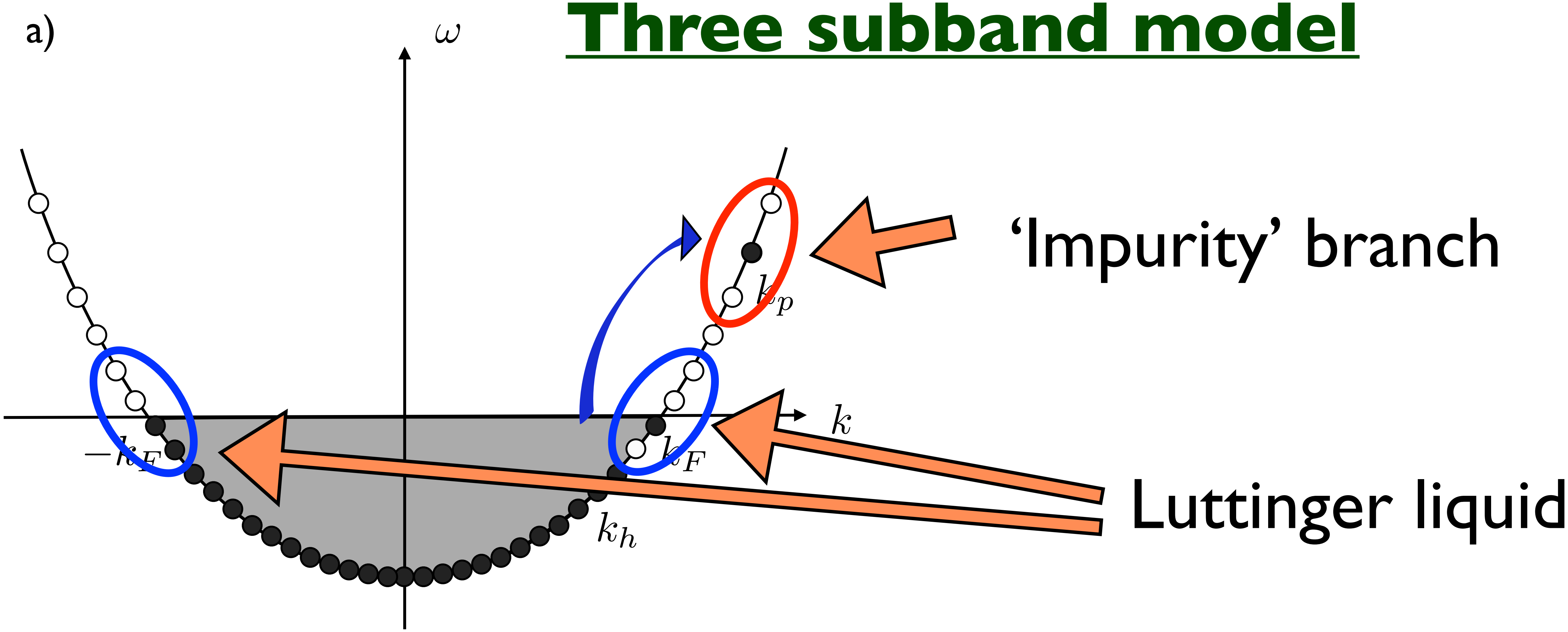
around here?
(where response is large)

Dynamical correlators: Nonlinear Luttinger Liquid Theory

Glazman, Imambekov, Khodas, Kamenev, Cheianov, Pustilnik, Affleck, Pereira, Sirker, JSC, ...

Observation: acting on ground state, an operator creates (few) high-energy + (many) low-energy excitations

Three subband model

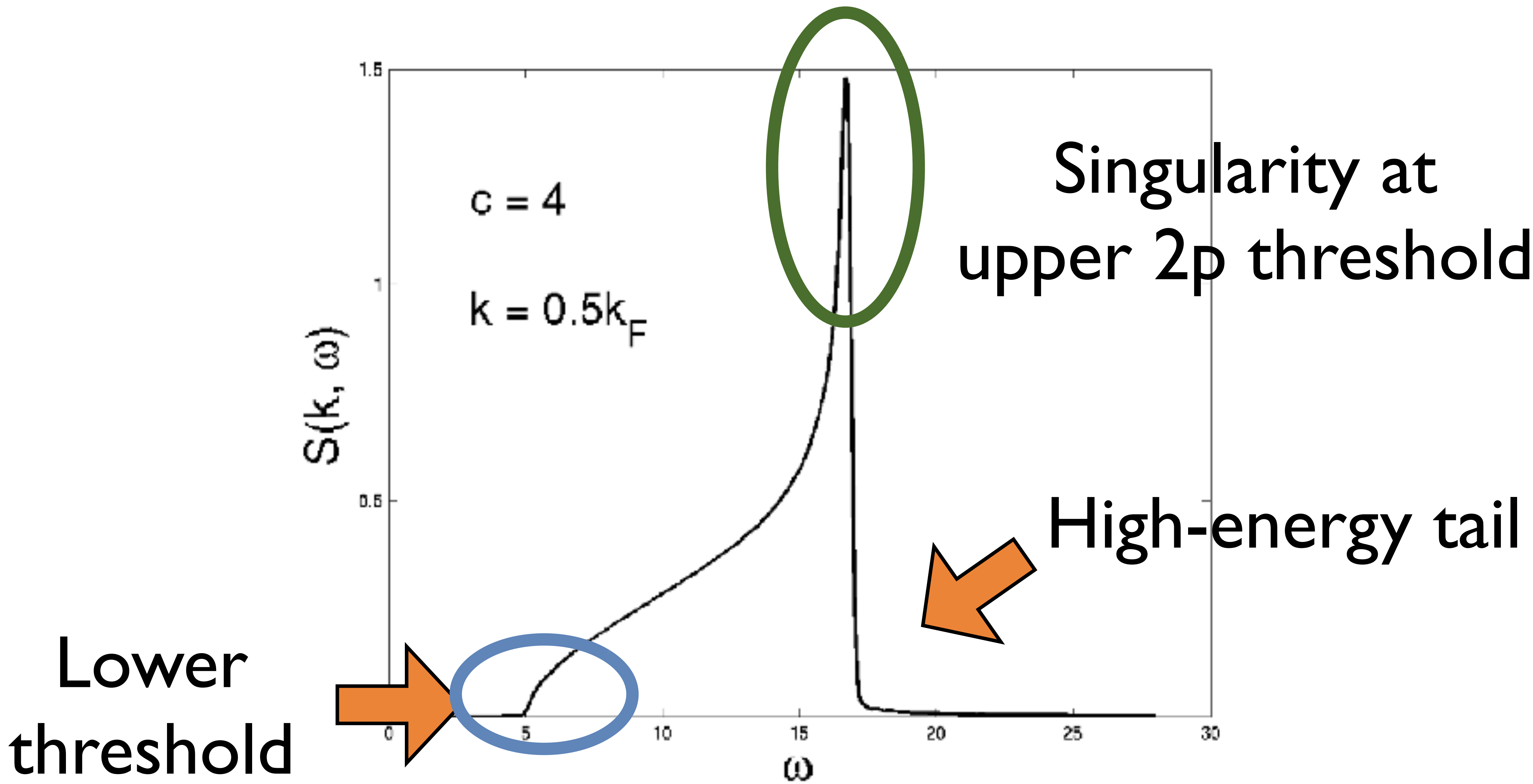


Singularity structure of response functions

(Khodas, Pustilnik, Kamenev, Glazman; Imambekov & Glazman)

Dynamical structure factor for interacting bosons

$$S \sim \frac{1}{|\omega - \epsilon_I|^{\mu_1}} (\theta(\epsilon_I - \omega) + \nu_1 \theta(\omega - \epsilon_I))$$

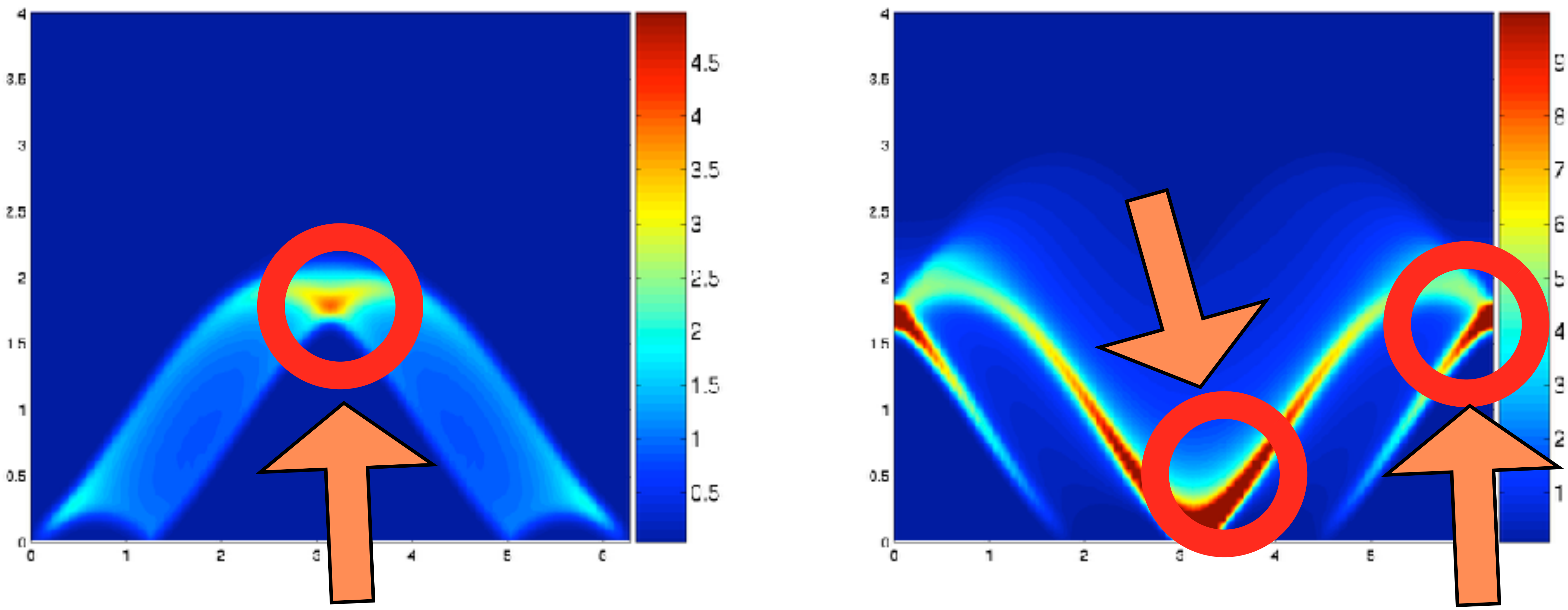


$$S \sim (\omega - \epsilon_{II})^{\mu_2}$$

Adilet Imambekov
1982-2012

Now: correlations, everywhere they matter?

$S^{zz}(k,\omega)$ $S^{+-}(k,\omega)$



Fact: at incommensurate fields, the correlation is mathematically *nonvanishing everywhere.*

Conjecture: *it is also non-smooth everywhere*

(there are infinitely many intercrossing thresholds).

$$\forall(k,\omega), \exists\ n < \infty \mid \partial^n_{k,\omega} S(k,\omega) \text{ does not exist}$$

Dynamics
Far from the
Ground State

Why be content with ground states? Open up the sea!

left Fermi sea

right Fermi sea

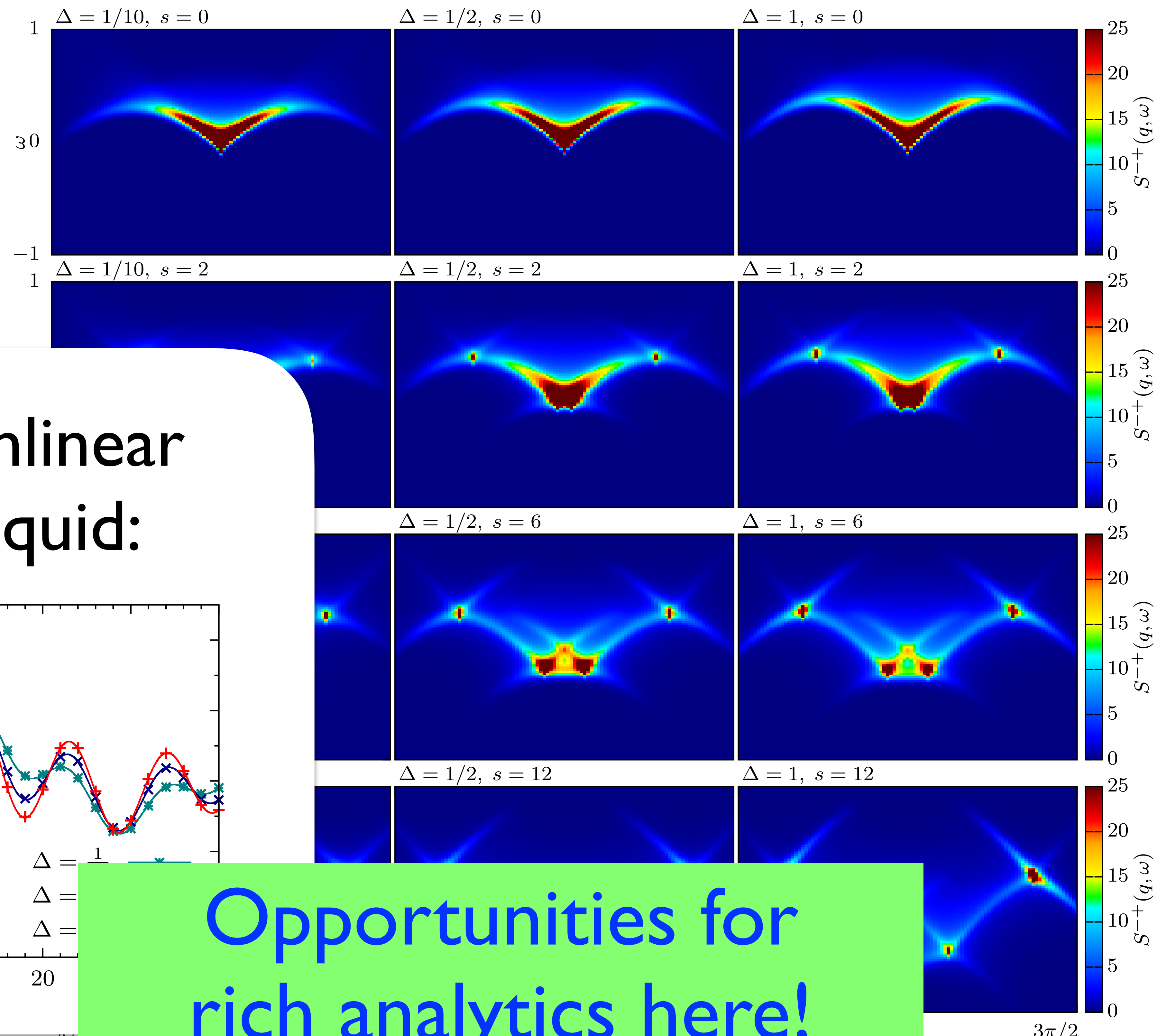
'Moses state'

Moses state in XXZ

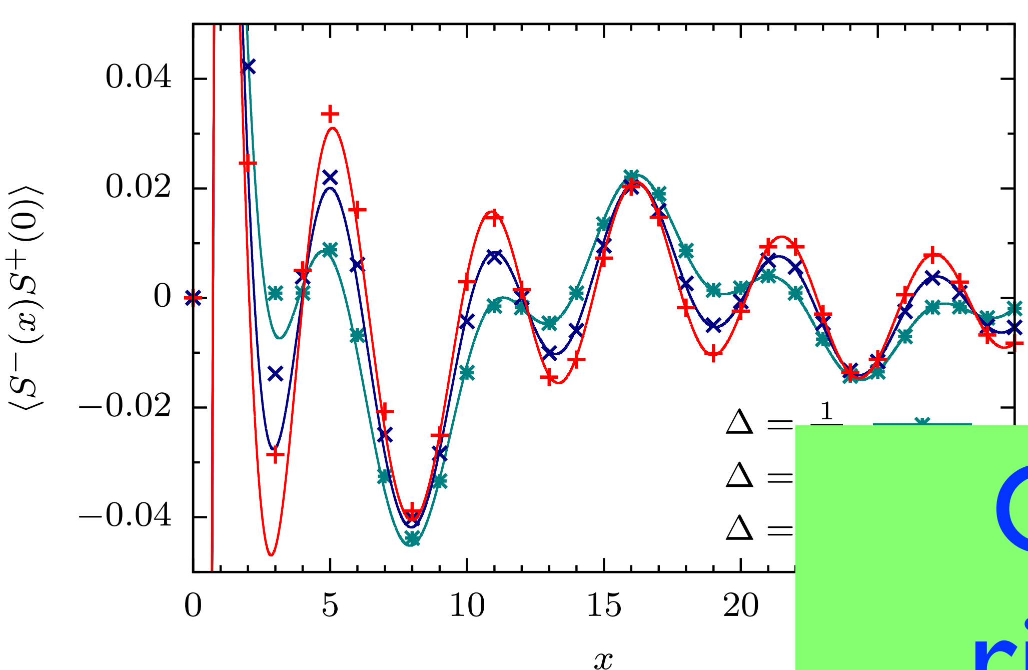
Vlijm, Eliëns, Caux, SciPost Phys. 1, 008 (2016)

$S^{-+}(k,\omega)$

as a function of splitting



Fits with nonlinear Luttinger liquid:



Opportunities for rich analytics here!

Integrable Models Beyond Equilibrium

Out-of-equilibrium using integrability

Pulsed:

- The super Tonks-Girardeau gas
- Split Fermi seas (Moses states)
- Spin echo in quantum dots
- Quasisolitons

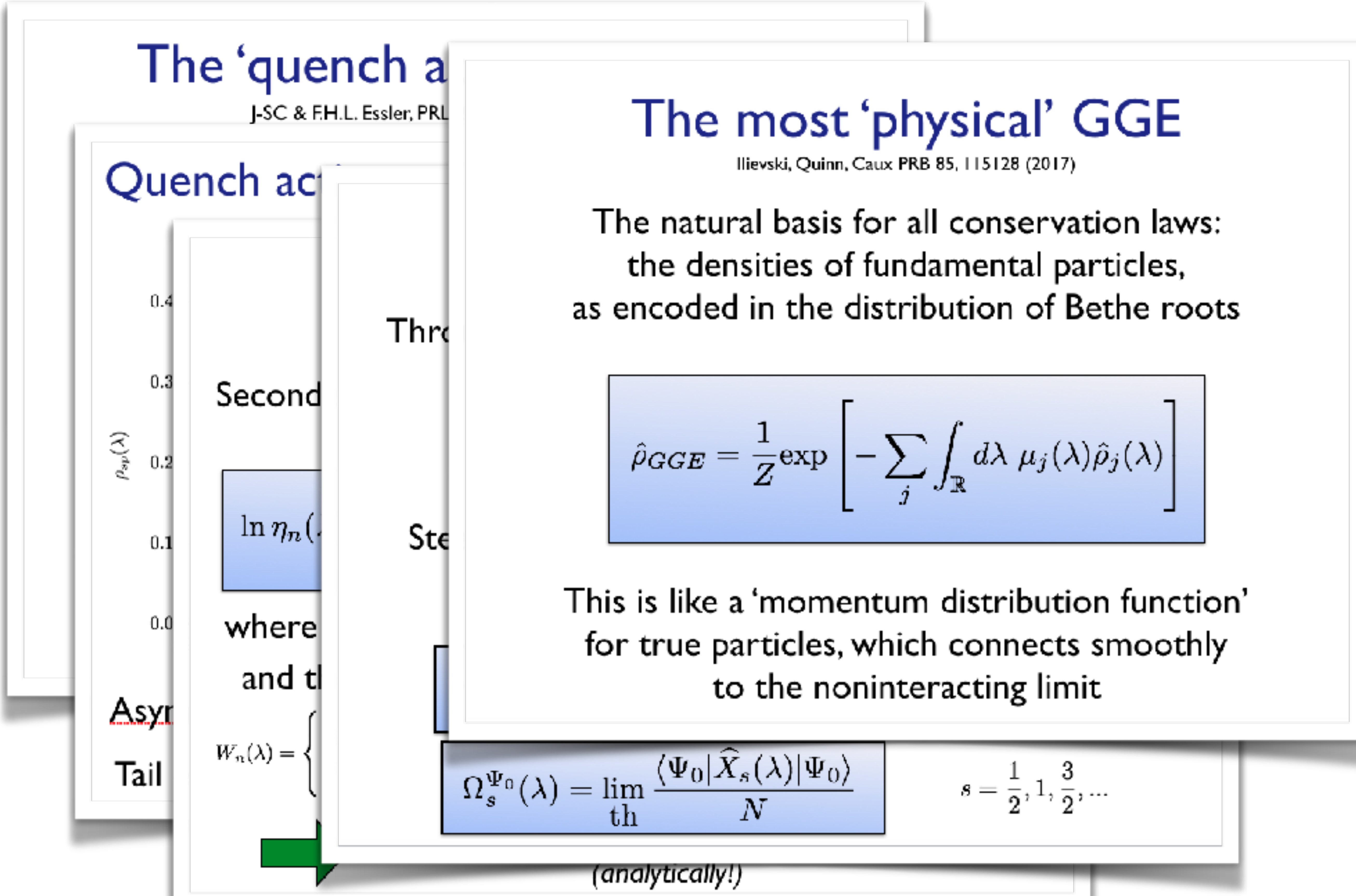
Quenched:

- Interaction quench in Richardson
- Domain wall release in Heisenberg
- Geometric quench
- Interaction cutoff in Lieb-Liniger
- Release of trapped Lieb-Liniger
- BEC to Lieb-Liniger quench
- Quantum Newton's Cradle in TG
- Néel to XXZ quench
- Generalized hydrodynamics

Driven:

- Floquet driving central spin
- Floquet driving Heisenberg

Progress on quenches



Quasisoliton dynamics in spin chains

Solitons (classical)

John Scott Russell:
solitary wave of translation (1834)

(Herriot-Watt University)

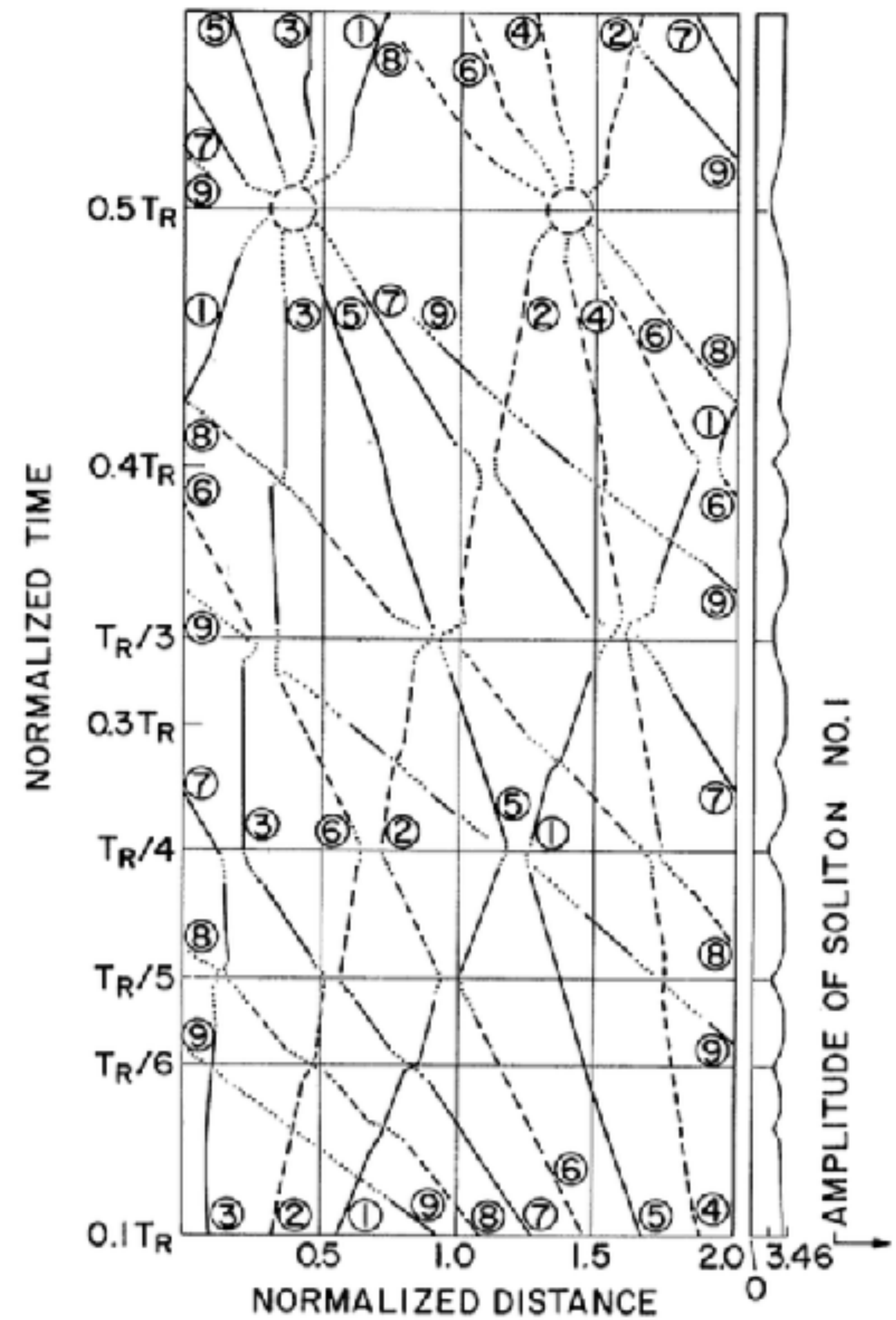
Solitons (classical)

(**Boussinesq**)
Korteweg-de Vries
equation

$$\partial_t u + u\partial_x u + \delta^2 \partial_x^3 u = 0$$

First simulations:
Fermi-Pasta-Ulam-(Tsingou)
absence of ergodicity
Further simulations:
Zabusky & Kruskal 1965
concept of a soliton

<u>**Classical inverse scattering**</u>

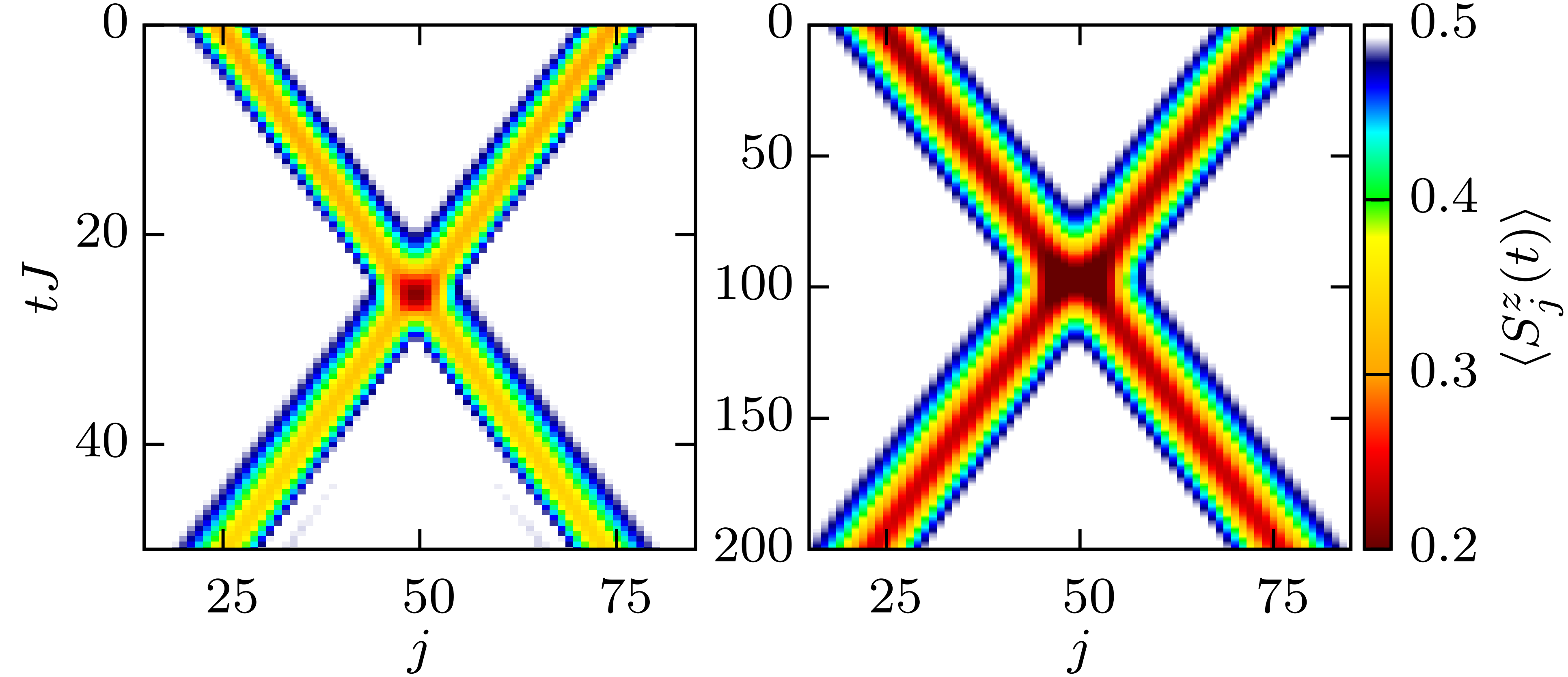


Quasisoliton scattering (quantum)

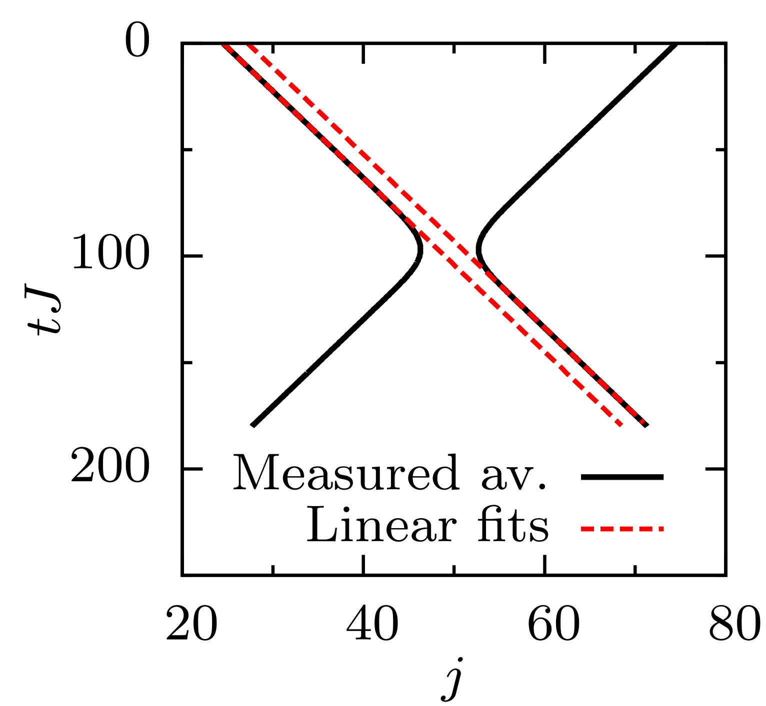
Vlijm, Ganahl, Fioretto, Brockmann, Haque, Evertz and Caux, 2015

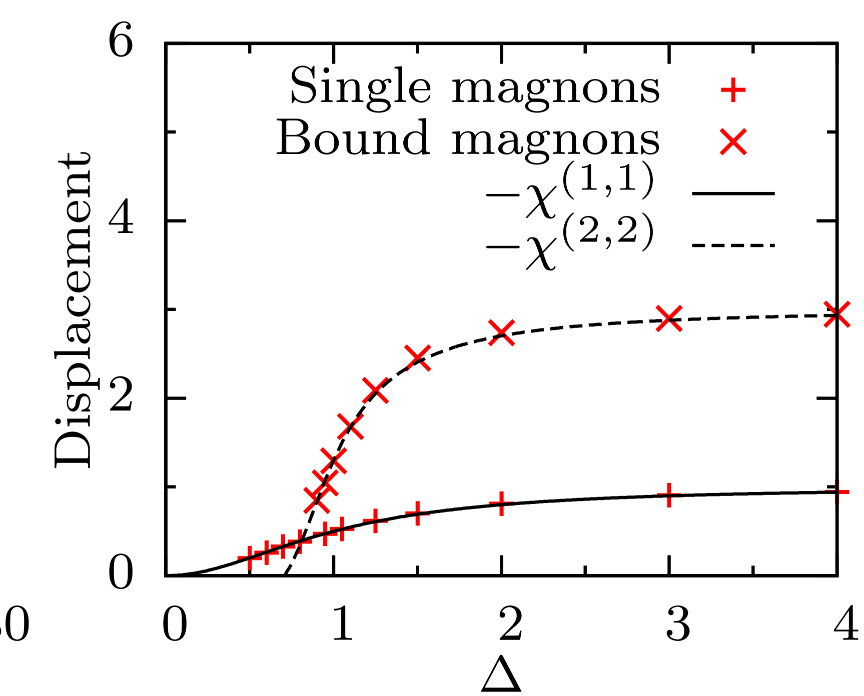
'Worldlines' of colliding wavepackets:

$\Delta = 2$



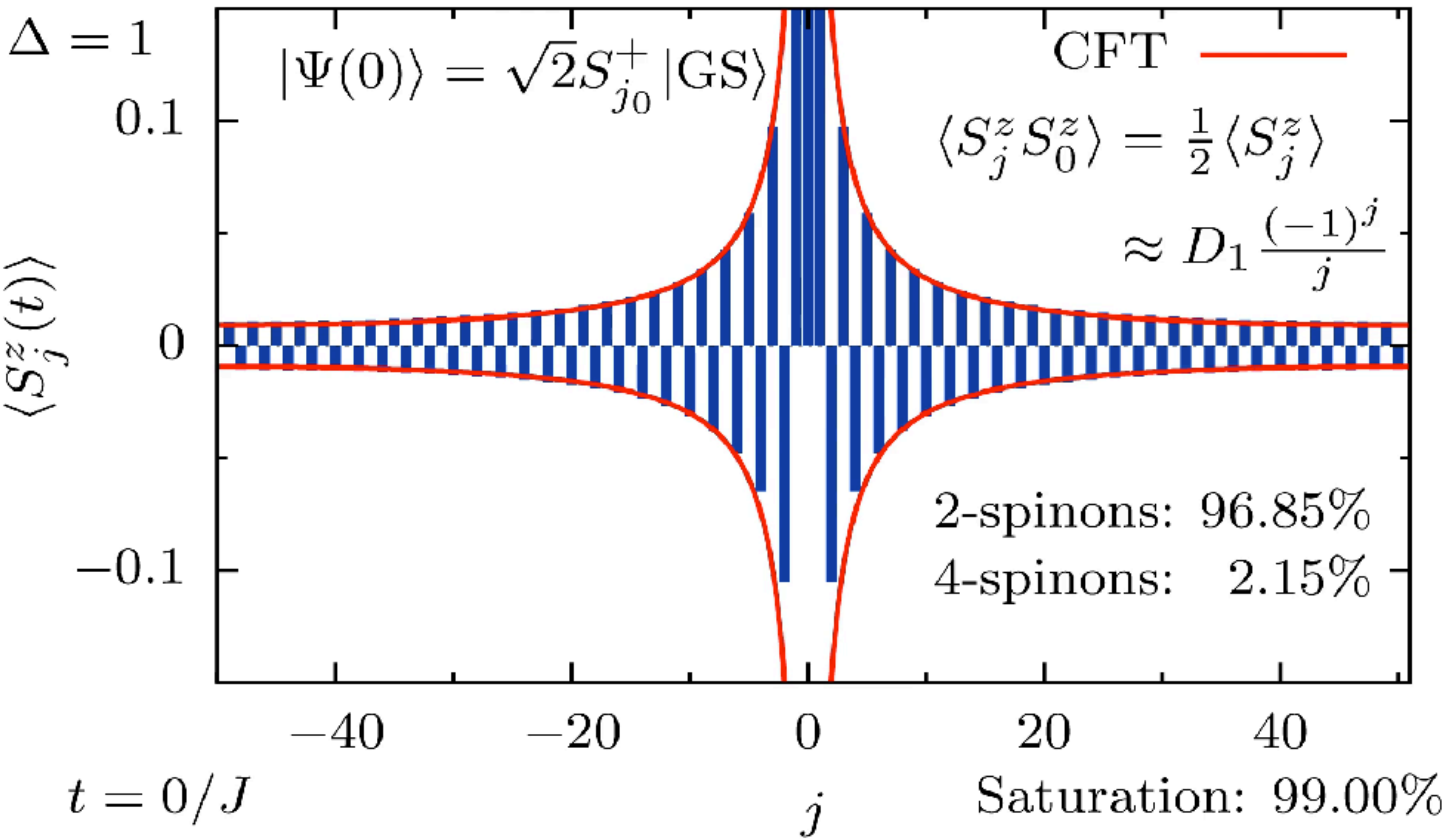
Displacement as a function of anisotropy (fixed incoming momenta)





Spinon dynamics in real space/time

Vlijm, Caux, PRB 2016



In memoriam

Ludvig Dmitrievich Faddeev
23/3/1934 - 26/2/2017

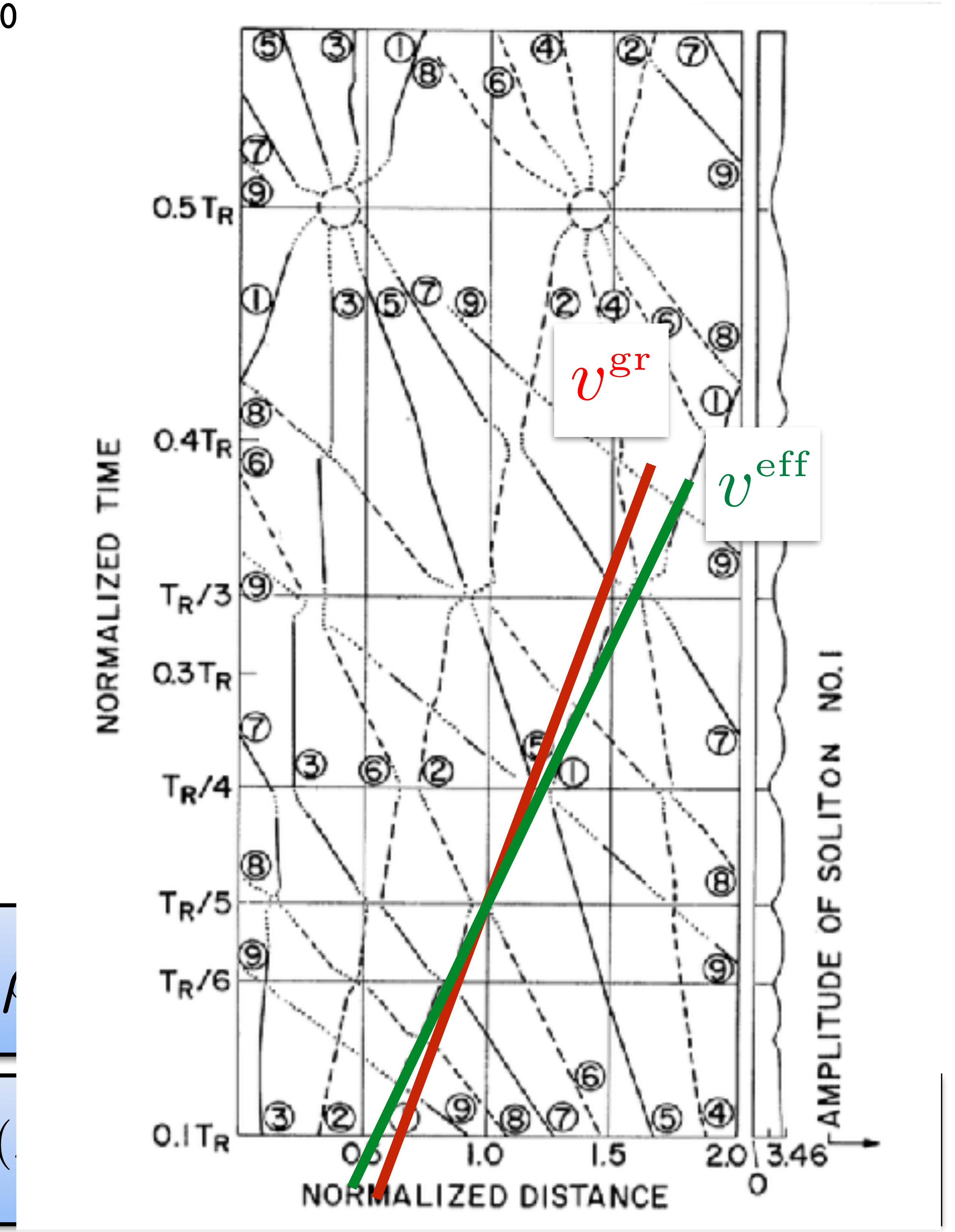
Generalized hydrodynamics

Generalized hydrodynamics (GHD)

B. Bertini, M. Collura, J. De Nardis and M. Fagotti, PRL 117, 207201 (2016)
O.A. Castro-Alvaredo, B. Doyon and T. Yoshimura, PRX 6, 041065 (2016)
B. Doyon and T. Yoshimura, SciPost Phys. 2, 0

Quench from spatially inhomogeneous state

After initial dephasings: 'hydrodynamic' evolution described by local GGE



Two-equation summary:

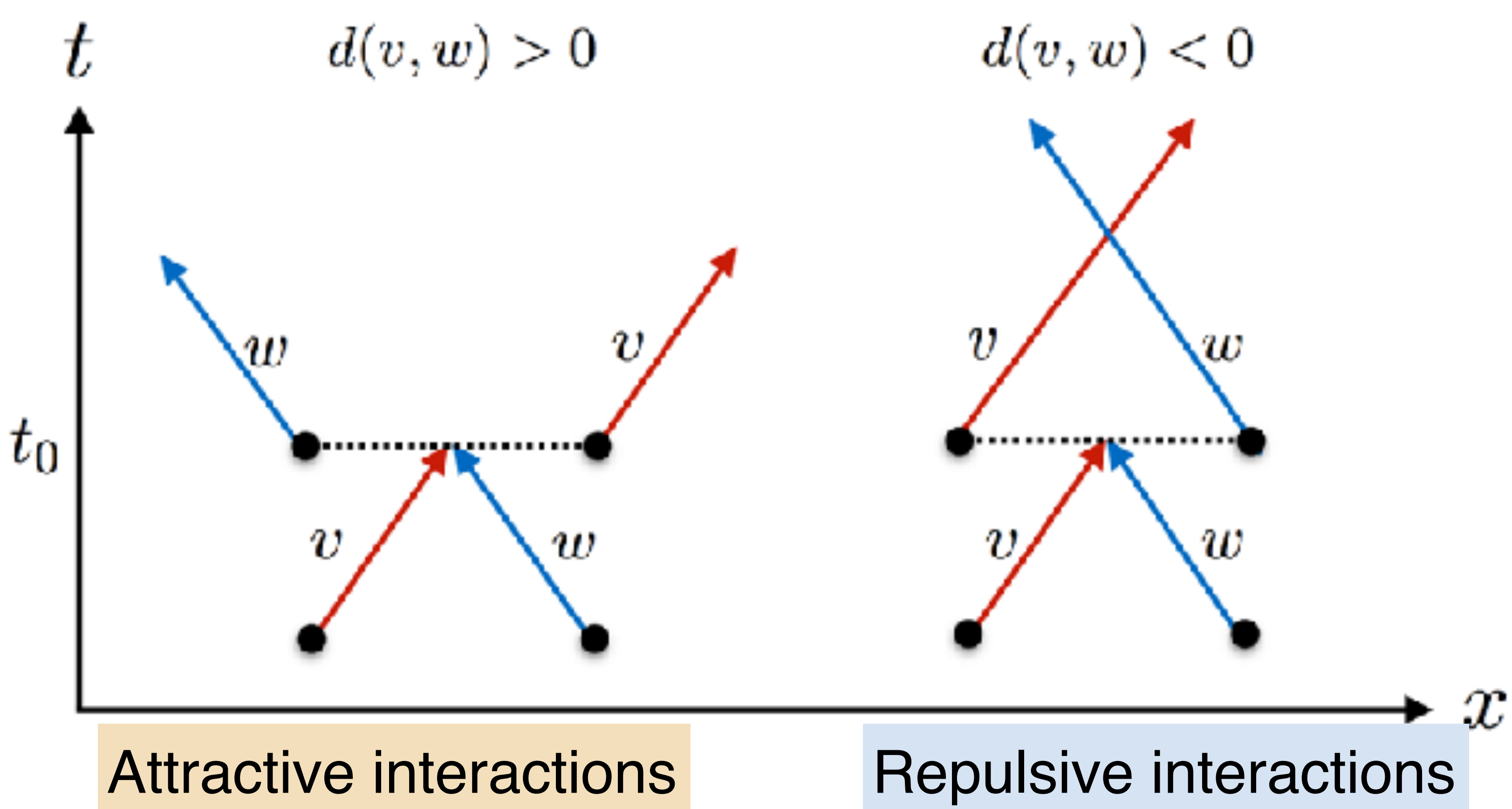
Local continuity equation (in terms of Bethe root densities) $\partial_t \rho$

Local effective velocity $v^{\mathrm{eff}}(\lambda) = v^{\mathrm{gr}}($

GHD as 'molecular dynamics': the flea gas

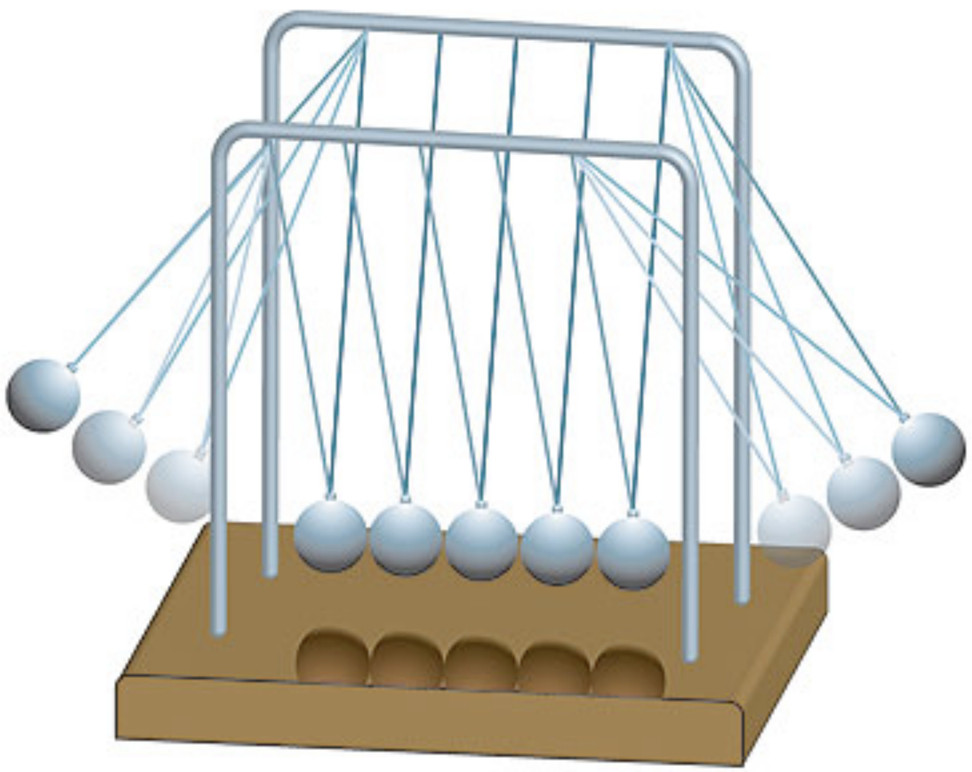
B. Doyon, T. Yoshimura and JSC, arXiv:1704.05482

- Encode initial state as a gas of quasisolitons
- Loop: evolve, collide and scatter
  (as if quasisolitons were classical particles, using displacement calculated from quantum phase shifts)

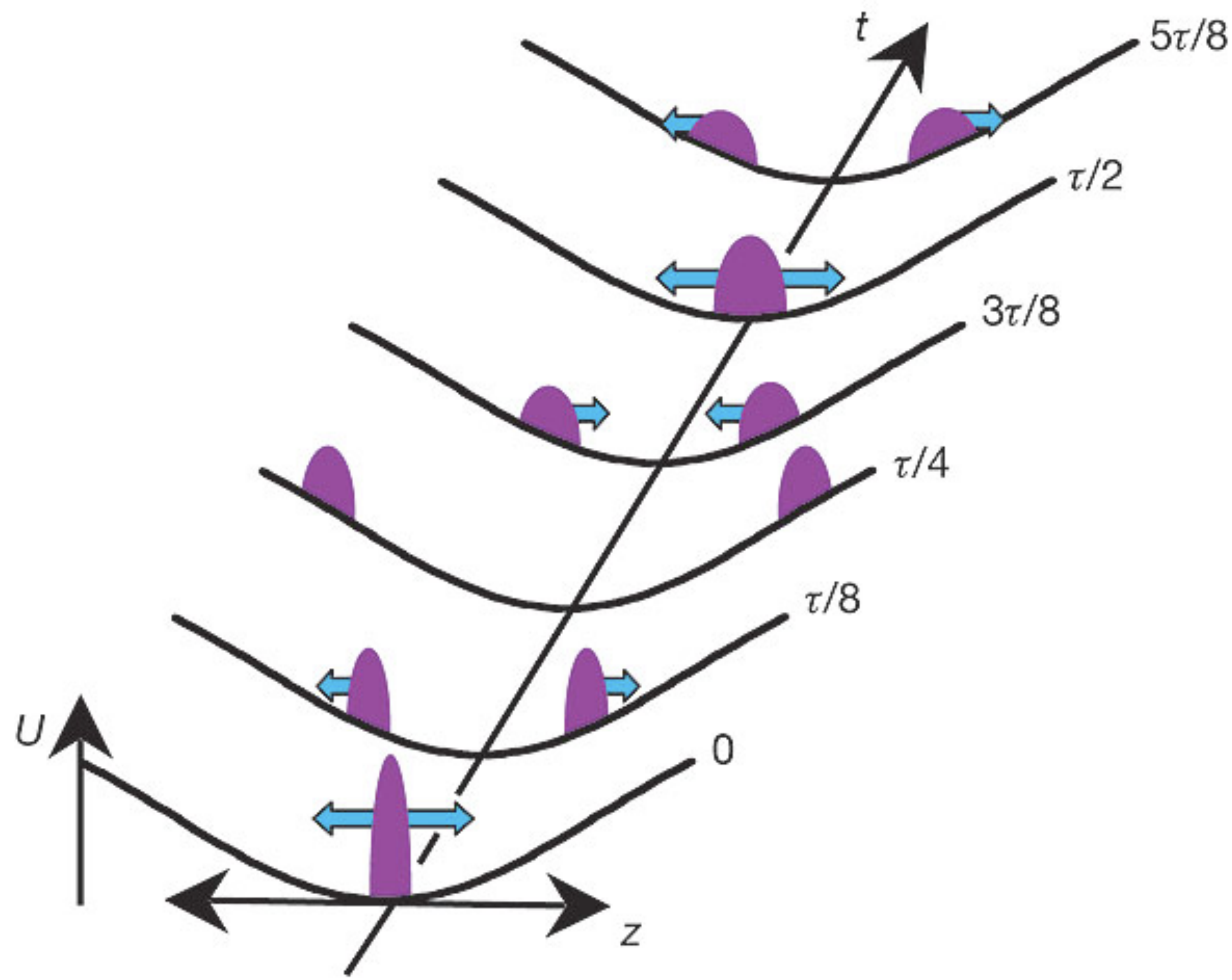


As simple as it gets in the integrability business!

David Weiss's quantum Newton's cradle experiment



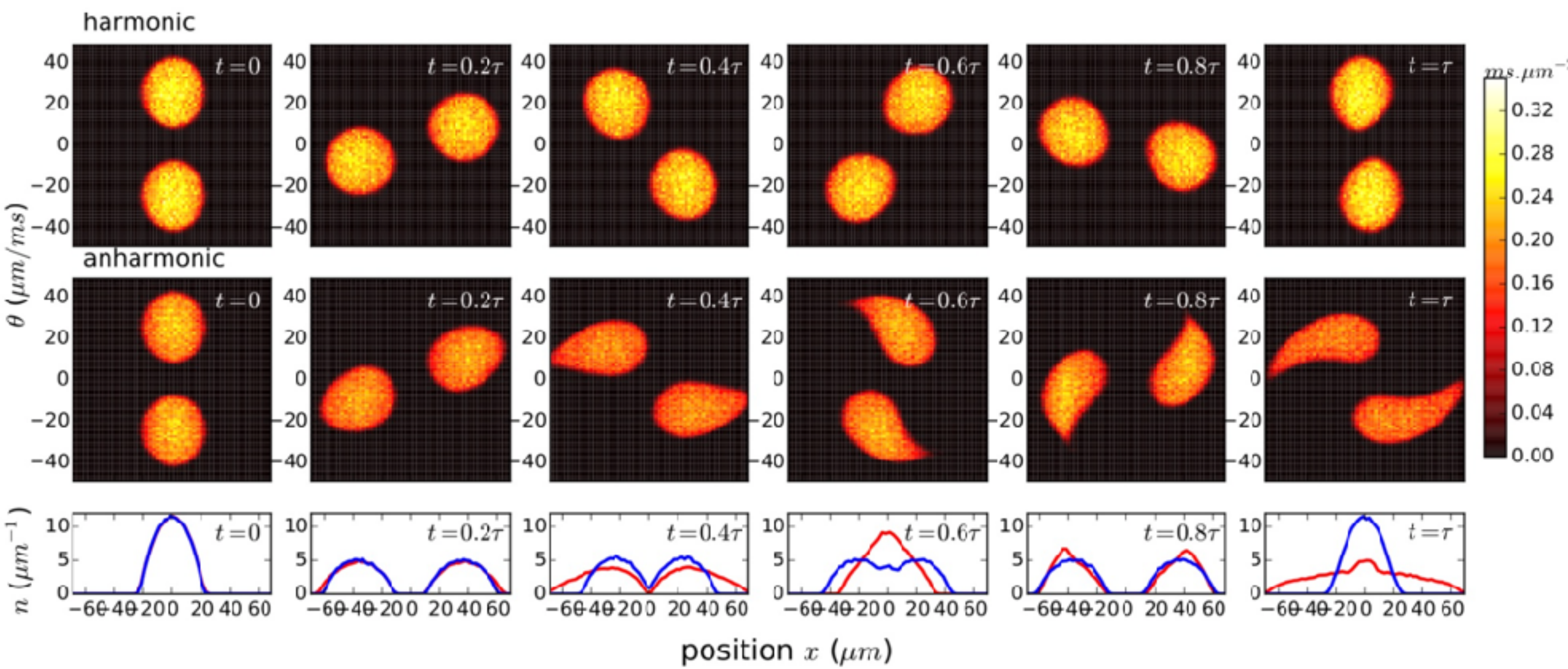
Ergodicity (or lack thereof) in interacting quantum systems close to an integrable model



The flea gas in a force field: simulating the quantum Newton's cradle

JSC, B. Doyon, J Dubail, R. Konik and T. Yoshimura, to appear

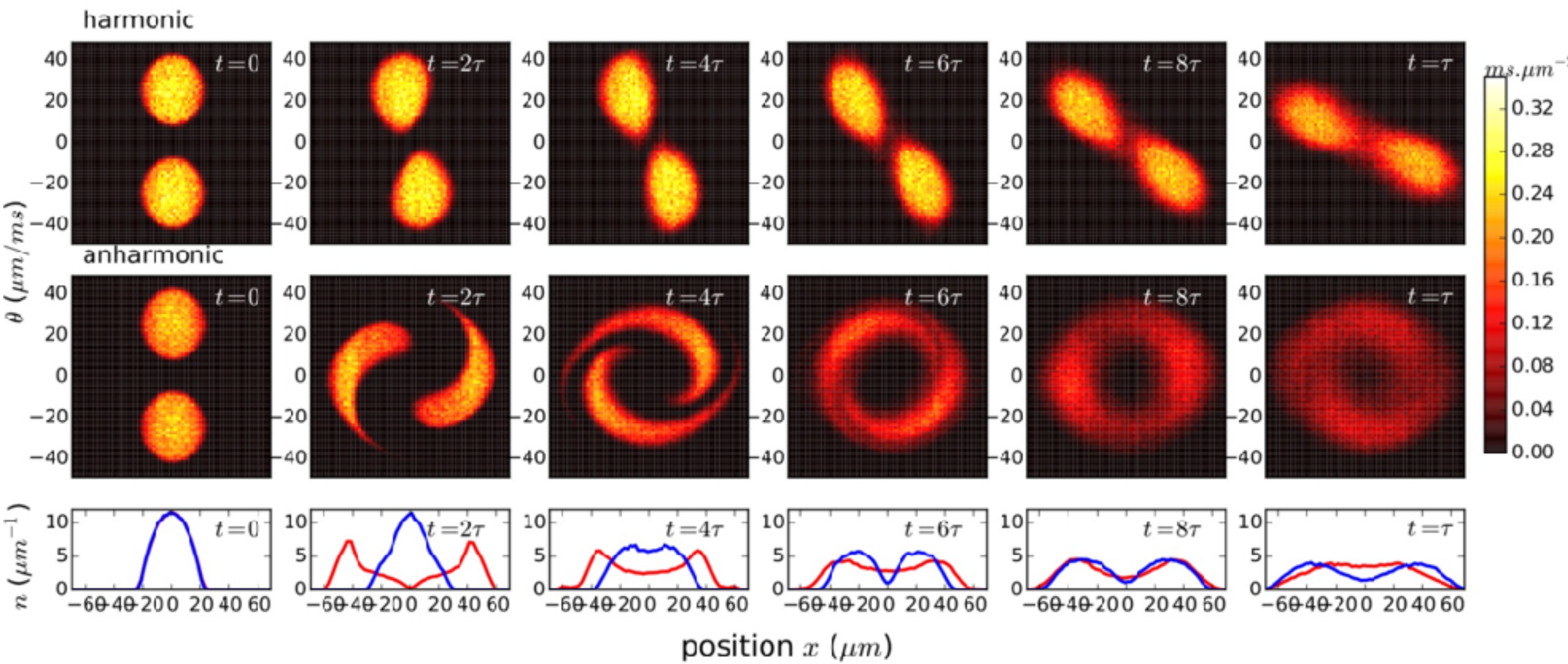
'Oscillation'-like dynamics at short time scales



The flea gas in a force field: simulating the quantum Newton's cradle

JSC, B. Doyon, J Dubail, R. Konik, and T. Yoshimura, to appear

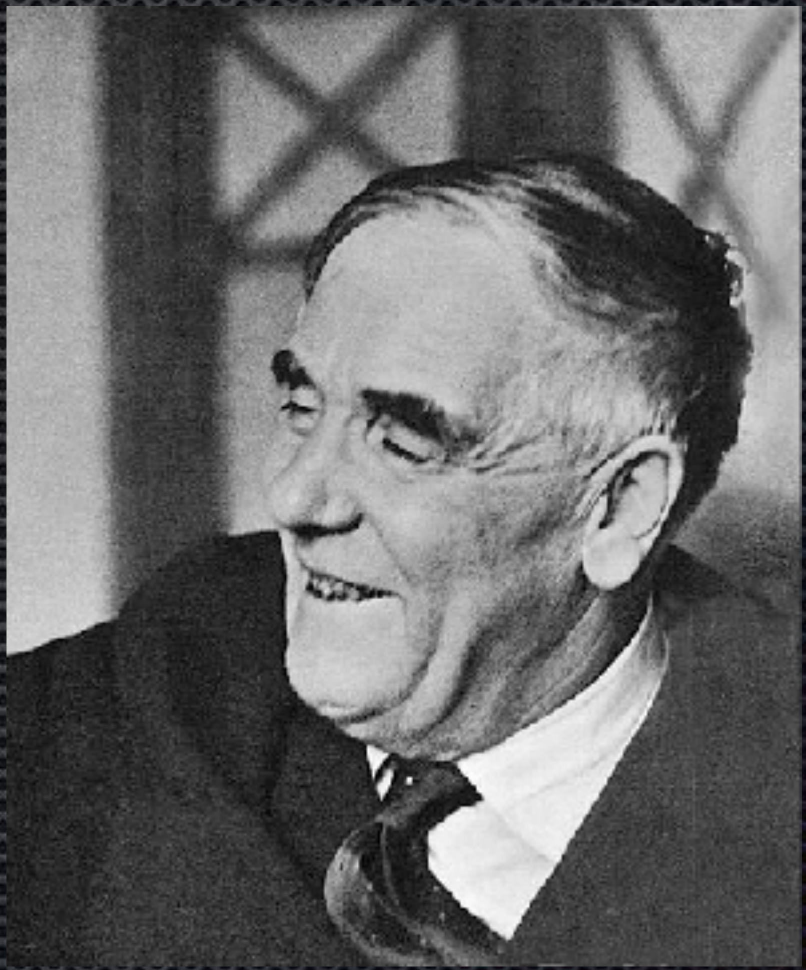
'Relaxation'-like dynamics at long time scales



Floquet dynamics

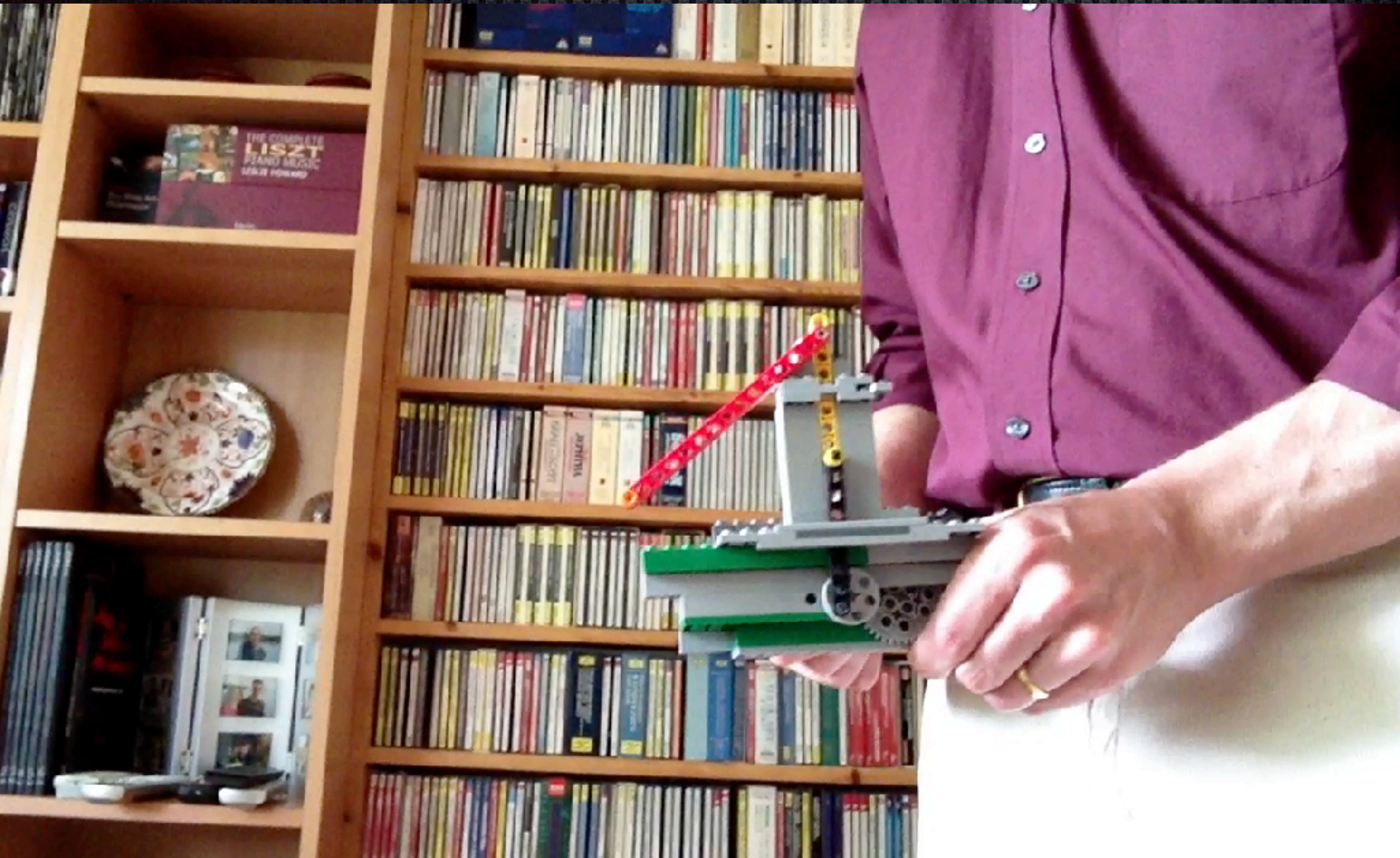
The simple pendulum on its head

Kapitza pendulum, 1951



Pyotr L. Kapitza
(8/7/1894-8/4/1984)

The Kapitza pendulum



Floquet basics

Unitary time evolution operator under action of periodic Hamiltonian $\hat{H}(t) = \hat{H}(t+T)$

$$\hat{U}(t) = T(e^{-i\int_0^t dt' H(t')}) = \hat{P}(t) e^{-i\hat{H}_F t}$$

"fast motion" operator

$$\hat{P}(t+T) = \hat{P}(t)$$

$$\hat{P}(nT) = \mathbf{1} \ (\mathbf{n} \in \mathbb{Z})$$

Floquet Hamiltonian

Stroboscopic Floquet operator $\hat{U}_F \equiv \hat{U}(T) = e^{-i\hat{H}_F T}$

Diag'n: $\hat{H}_F = \sum_n \epsilon_n |\phi_n\rangle\langle\phi_n|$ $\quad \hat{U}_F = \sum_n e^{-i\theta_n} |\phi_n\rangle\langle\phi_n|$

t evolution: $|\psi_n(t)\rangle = e^{-i\epsilon_n t}|\phi_n(t)\rangle, \quad |\phi_n(t)\rangle = \hat{P}(t)|\phi_n\rangle$

"Quench-dequench" Floquet protocol

Let's consider the simple case of periodically "switching" between two Hamiltonians:

$$\hat{H}(t) = \begin{cases} \hat{H}_1 & \text{for} \quad 0 < t < \eta T, \\ \hat{H}_2 & \text{for} \quad \eta T < t < T, \end{cases}$$

so $$\hat{U}_F \equiv e^{-i\hat{H}_F T} = e^{-i(1-\eta)T\hat{H}_2} e^{-i\eta T\hat{H}_1}$$

One then finds the nice identities

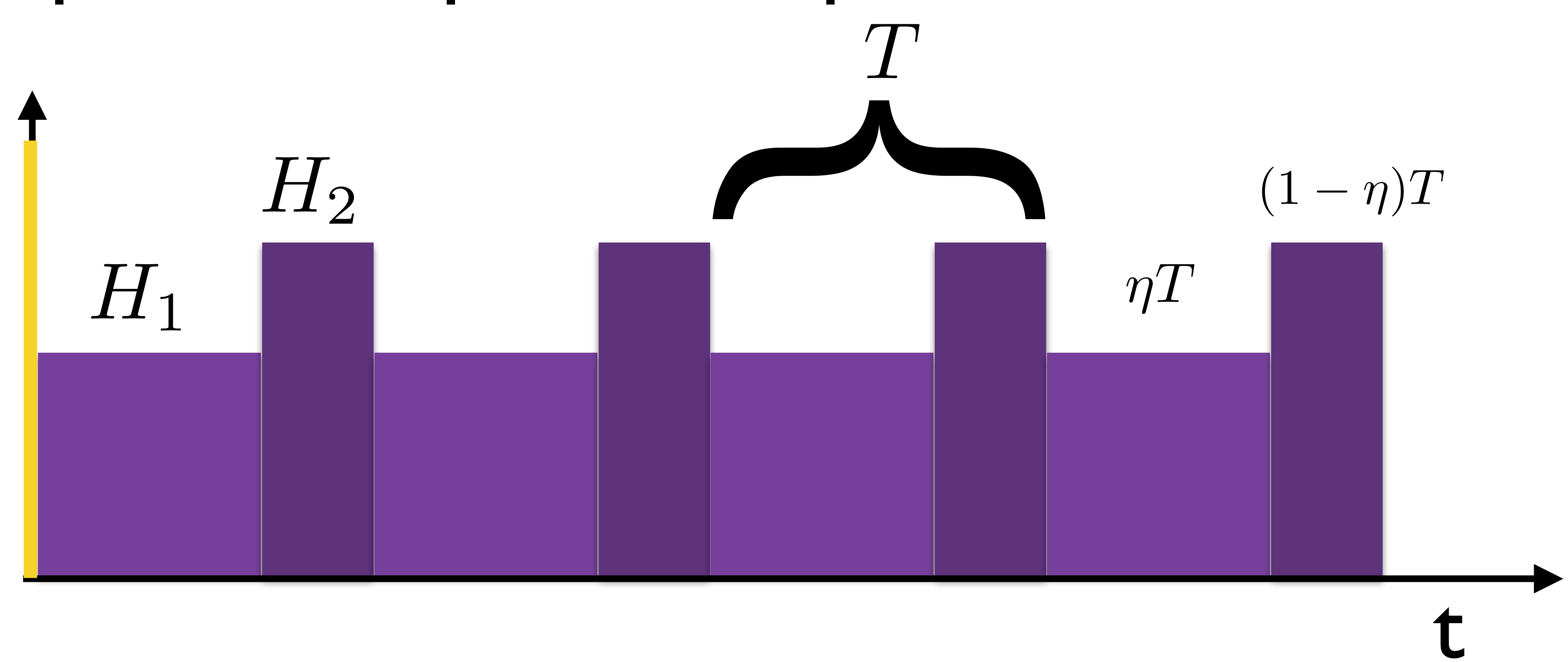
$$\epsilon_n = \frac{\theta_n}{T} = \langle\phi_n|\hat{H}_F|\phi_n\rangle \qquad \frac{\partial\theta_n}{\partial T} = \langle\phi_n|\hat{H}_{avg}|\phi_n\rangle$$

with t-avg Hamiltonian $\hat{H}_{Avg} = \eta\hat{H}_1 + (1-\eta)\hat{H}_2$

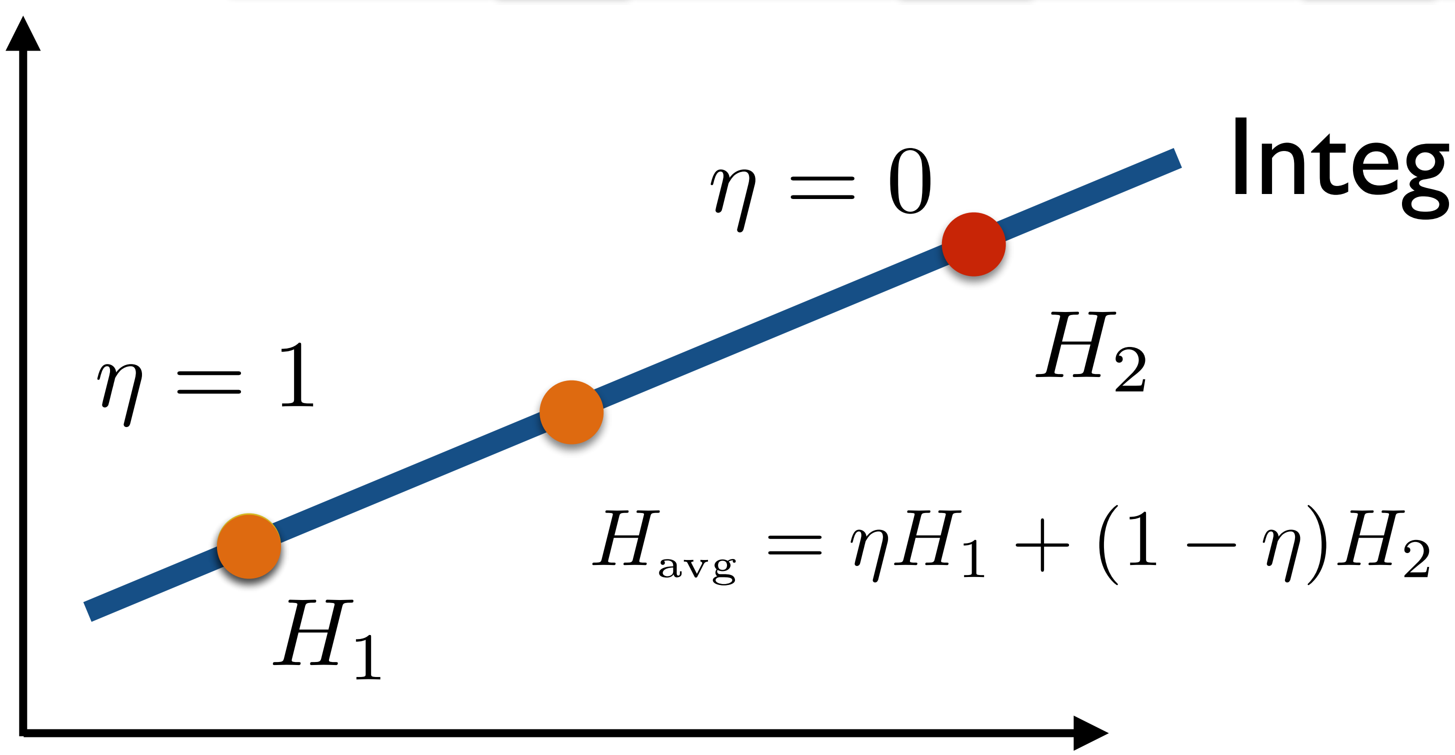
Floquet'ing integrable models

P. Claeys and JSC, arXiv:1708.07324

Idea:
- take a one-parameter family of integrable models
- do quench/dequench sequences on this manifold



Integrable manifold



Time-avg Hamiltonian also integrable

Floquet'ing integrable models

P. Claeys and JSC, arXiv:1708.07324

Hamiltonian: as you perhaps guessed, XXZ:

$$\hat{H}(t) = -J\sum_i \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta(t) S_i^z S_{i+1}^z\right]$$

Floquet protocol: binary switch between Δ_1 and Δ_2

Stroboscopic Floquet operator: $\hat{U}_F = e^{-i(1-\eta)T\hat{H}_2} e^{-i\eta T\hat{H}_1}$

No analytical solution, must rely on numerics

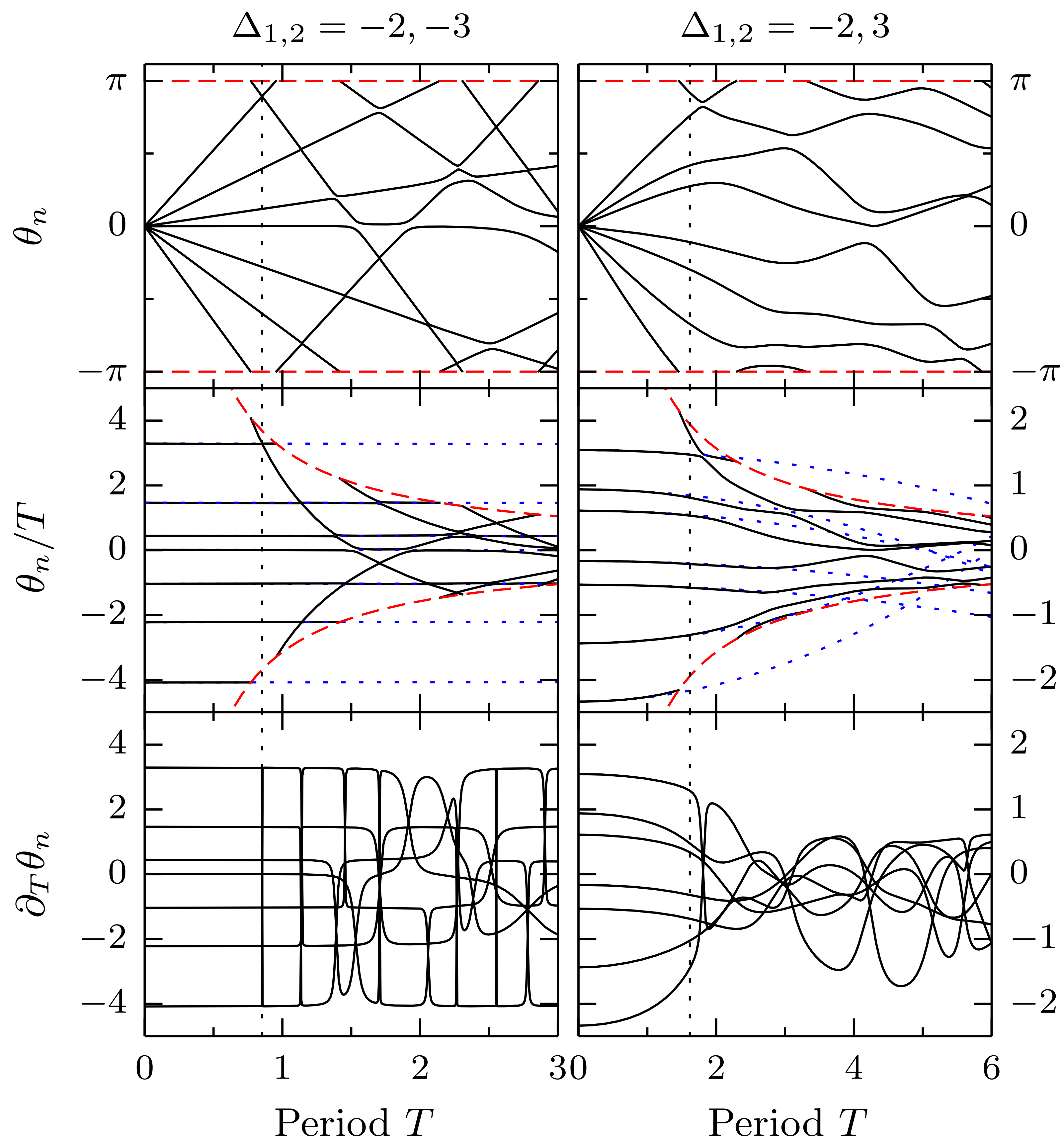
Focus on sector with $k = 0$ and $m_z = 1/3$

Floquet'ing integrable models

P. Claeys and JSC, arXiv:1708.07324

Floquet phases,
quasienergies
and
avg quasienergies
as a function of T

All crossings
are avoided



Eigenvalue statistics

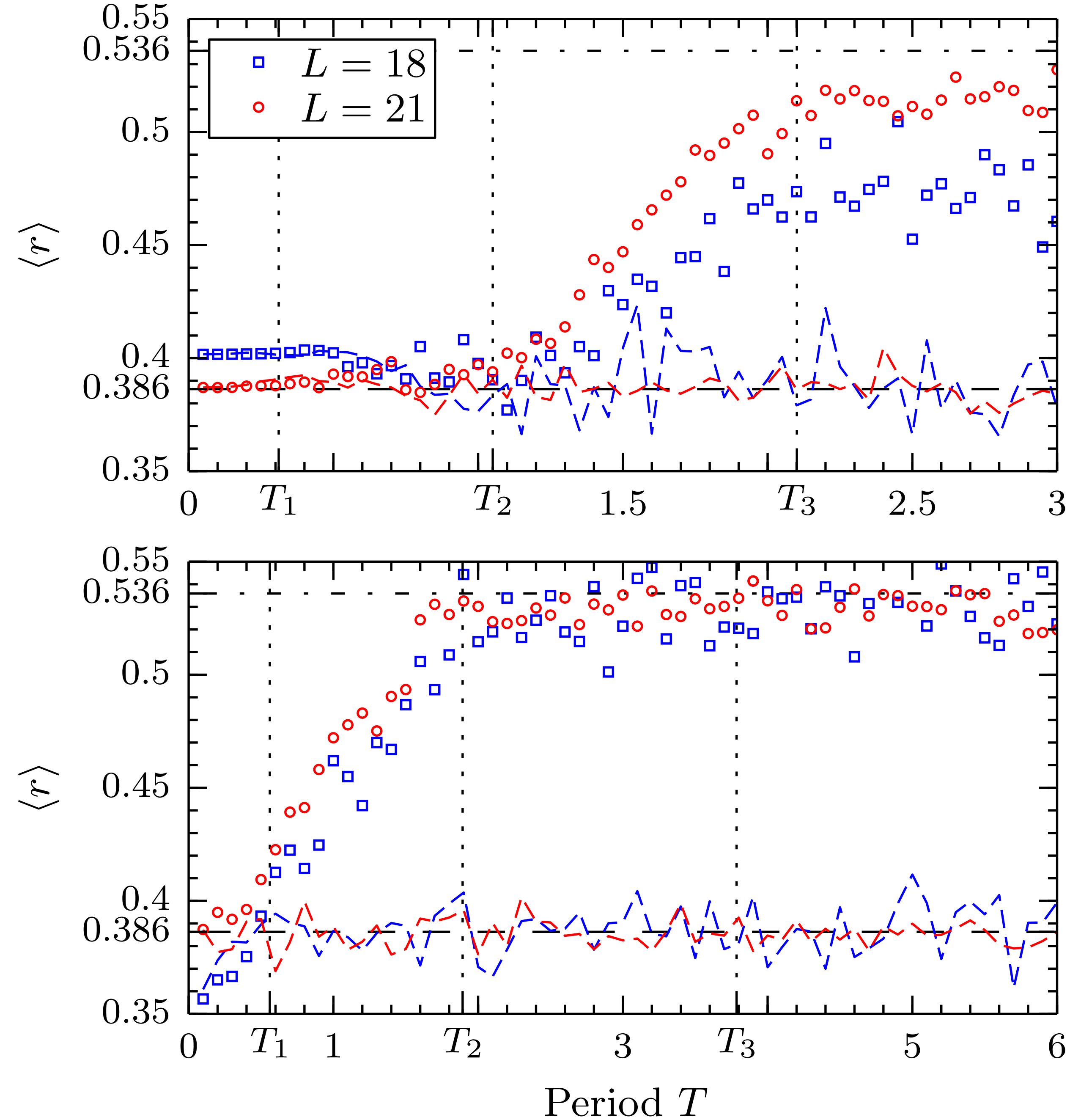
$$r = \frac{\min(s_n, s_{n+1})}{\max(s_n, s_{n+1})} \in [0,1], \;\; s_n = E_{n+1} - E_n$$

Expect $\langle r \rangle_{GOE} \approx 0.535989$ $\langle r \rangle_{POI} \approx 0.386295$

Quasienergies mark transition from Poisson (high frequency) to GOE (low freq)

$\langle r(\theta_n/T) \rangle$ (symbols)

$\langle r(\partial_T \theta_n) \rangle$ (dashed lines)



Integrability breaking in Richardson-Gaudin

A variational method for integrability breaking in Richardson-Gaudin models

P. Claeys, JSC, D. van Neck and S. De Baerdemacker, arXiv:1707.06793

States $$|\psi_{RG}\rangle = \prod_{\alpha=1}^{N} \left(\sum_{i=1}^{L} \frac{S_i^\dagger}{\epsilon_i - \lambda_\alpha} \right) |\downarrow \dots \downarrow\rangle$$

diagonalize RG systems $$\hat{H} = \sum_{i=1}^{L} \eta_i R_i, \qquad \eta_i \in \mathbb{R}$$

with $$R_i = S_i^0 + g \sum_{j \neq i}^{L} \frac{1}{\epsilon_i - \epsilon_j} \left[\frac{1}{2} \left(S_i^\dagger S_j + S_i S_j^\dagger \right) + S_i^0 S_j^0 \right]$$

provided $$1 + \frac{g}{2} \sum_{i=1}^{L} \frac{1}{\epsilon_i - \lambda_\alpha} - g \sum_{\beta \neq \alpha}^{N} \frac{1}{\lambda_\beta - \lambda_\alpha} = 0, \qquad \alpha = 1 \dots N$$

A variational method for integrability breaking in Richardson-Gaudin models

P. Claeys, JSC, D. van Neck and S. De Baerdemacker, arXiv:1707.06793

Central Spin model: $\hat{H}_{cs} = BS_1^z + g\sum_{i\neq 1}^{L}\frac{\vec{S}_1\cdot\vec{S}_i}{\epsilon_{0,1}-\epsilon_{0,i}}$

Perturbations: single spin, or two-spin terms

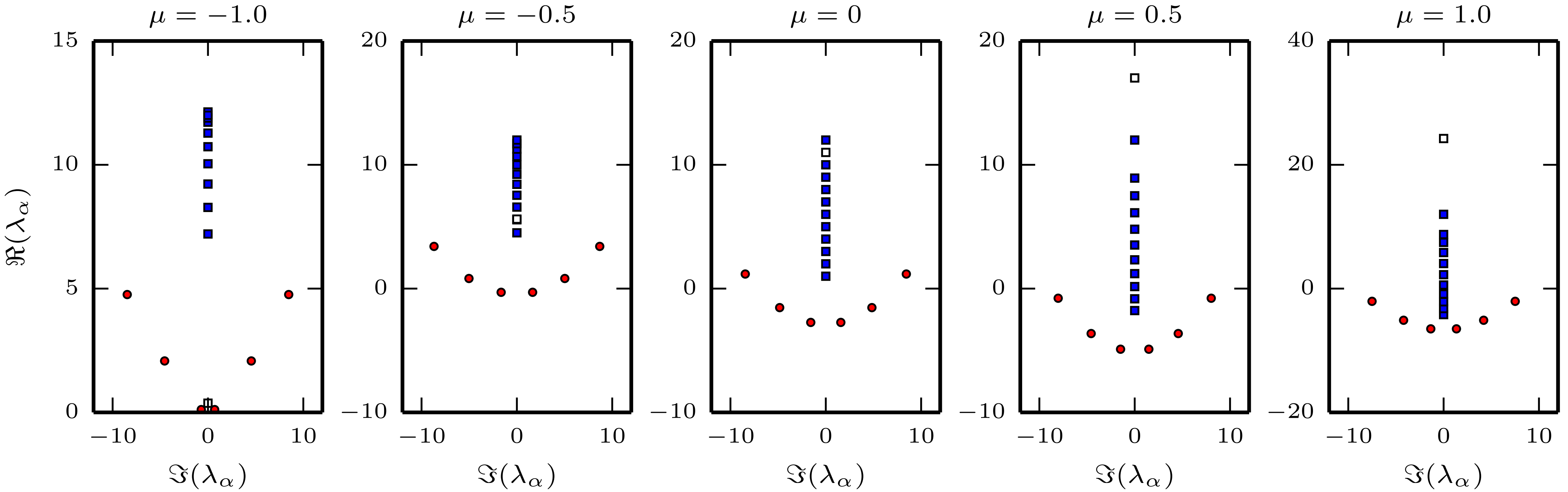
$$\hat{H} = \hat{H}_{cs} + \mu S_i^0, \qquad \hat{H} = \hat{H}_{cs} + \mu\vec{S}_i\cdot\vec{S}_j$$

Strategy: define a variational wavefunction using auxiliary quasi-energies as variational parameters, and optimize

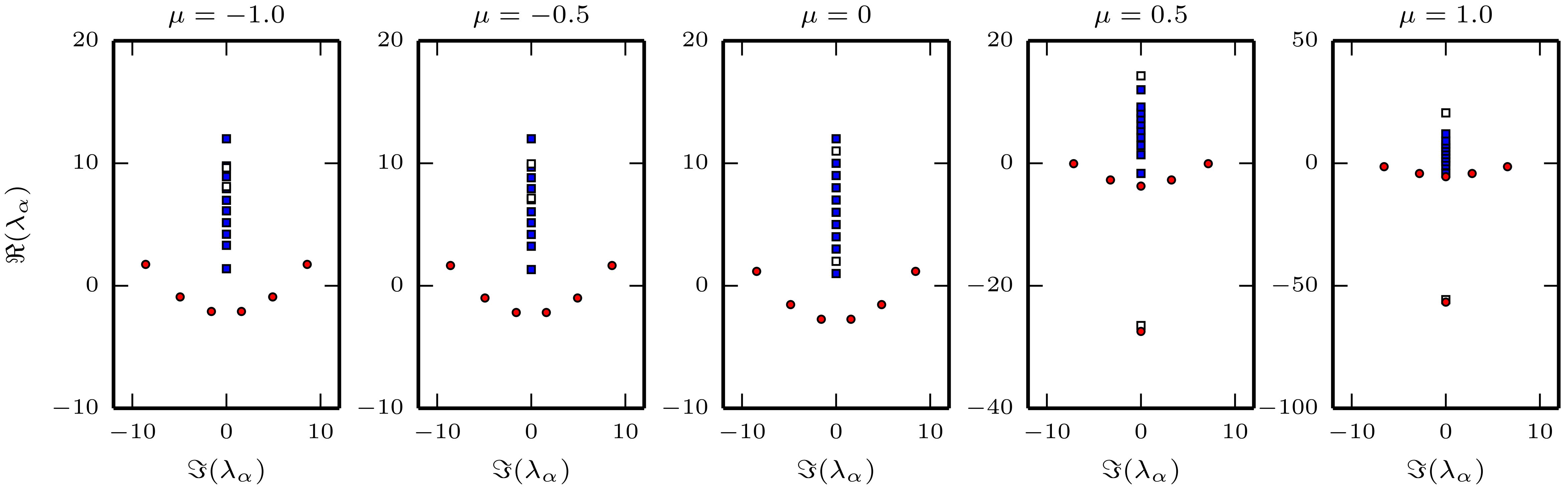
$$E\left[\psi_{RG}\right] = \frac{\langle\psi_{RG}|\hat{H}|\psi_{RG}\rangle}{\langle\psi_{RG}|\psi_{RG}\rangle}$$

Parameters of optimized states

Single spin perturbation:

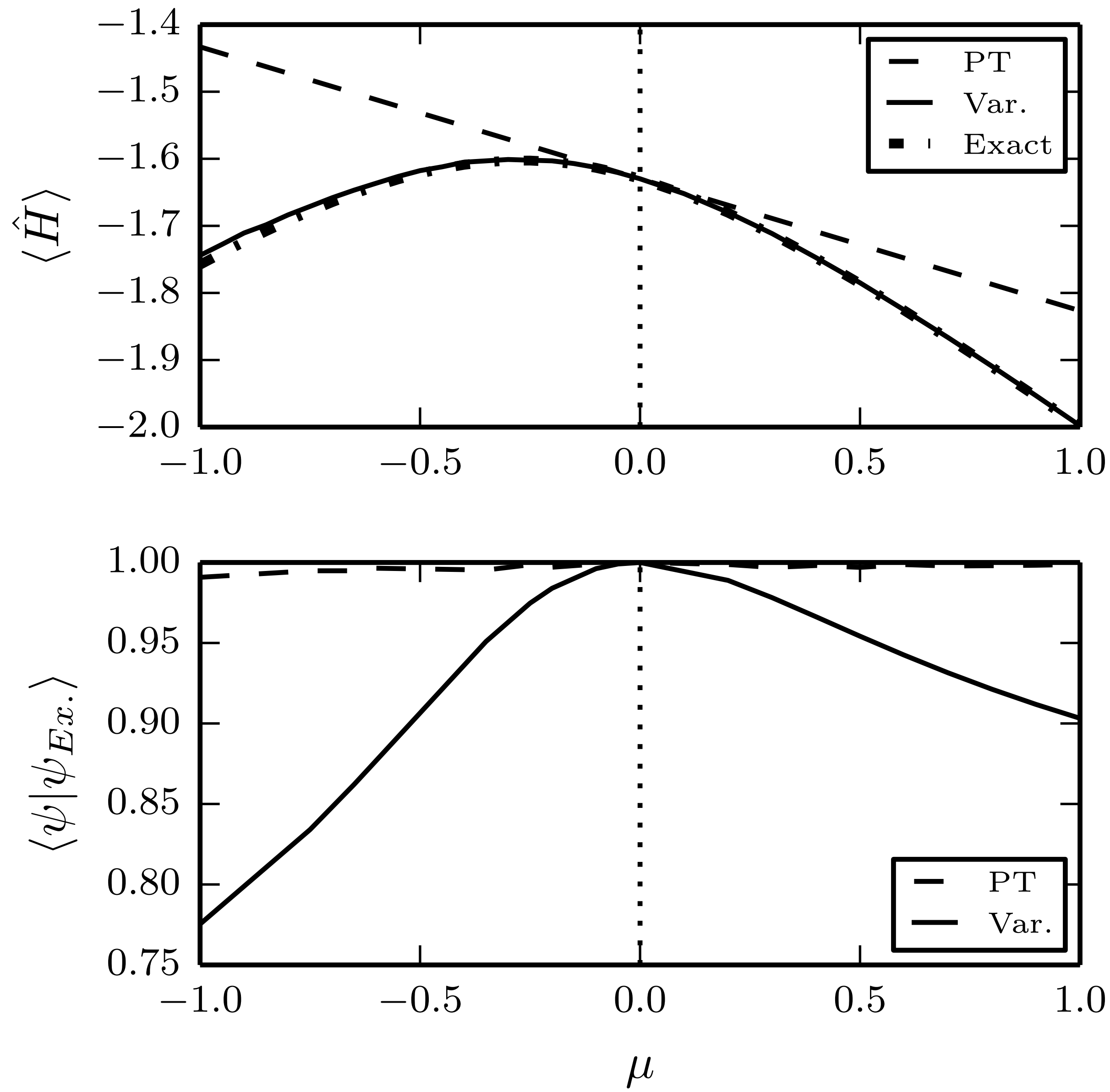


Two-spin perturbation:

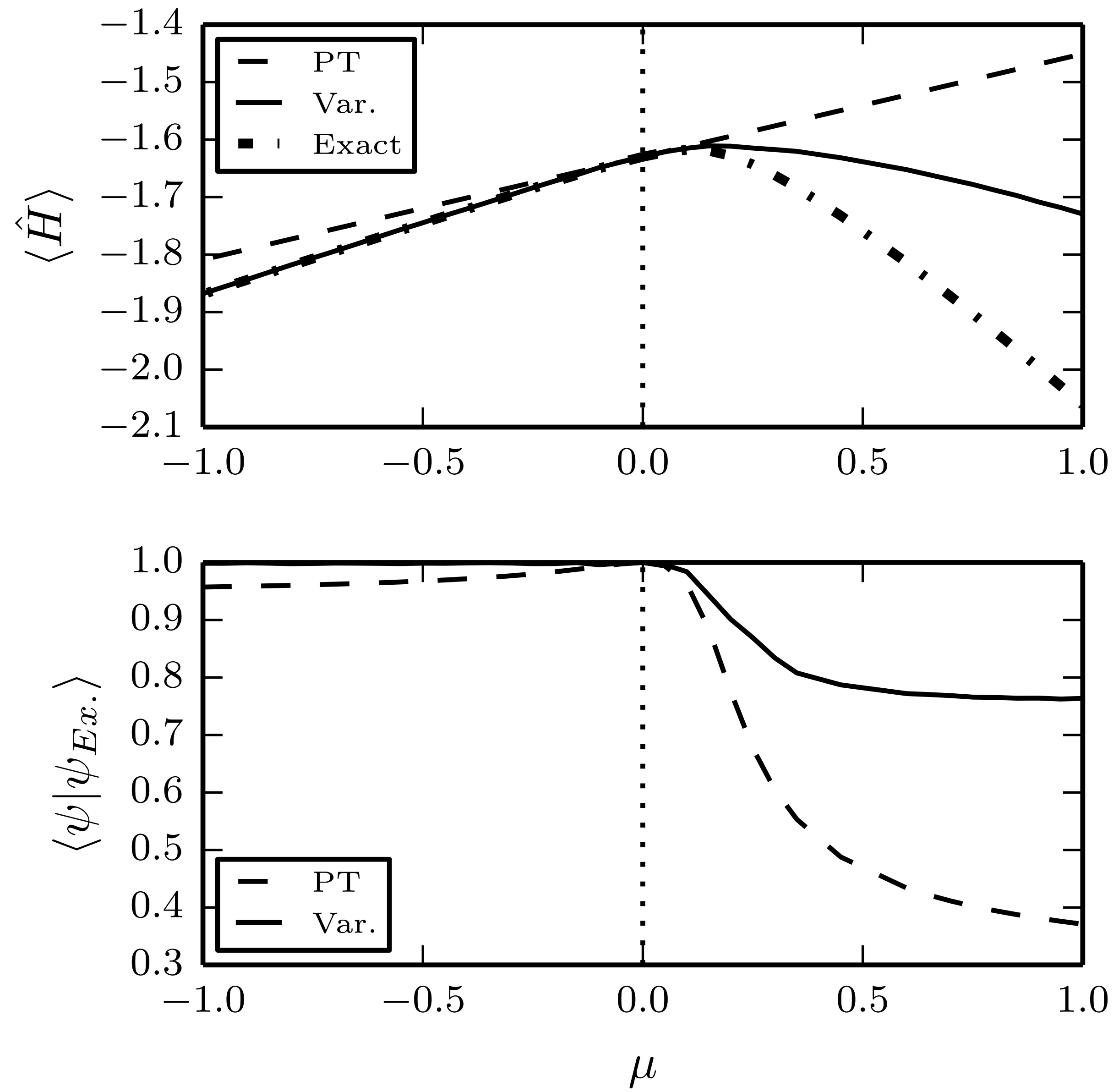


Energies of optimized states

Single spin perturbation:



Two-spin perturbation:




HAPPY
BIRTHDAY

Challenges for Jean Michel

- Compute XXZ gapless AFM dynamical correlations at finite field, beyond asymptotics

- Compute dynamical correlations on states other than ground states or thermal states (start: low-entropy ones like Moses states)

- Find exact overlaps of eigenstates of (pairs of) distinct Hamiltonians (and find other exact quench steady states using the Quench Action)

- Find integrable Floquet Hamiltonians

Conclusions

- Equilibrium dynamics in integrable systems
  - *Very healthy recent history*
- Quench dynamics in integrable systems
  - *Surprisingly healthy recent history!*
  - *Quench Action (no details presented here, look at papers!)*
  - *Generalized hydrodynamics: from quasisolitons to the flea gas*
- Floquet dynamics in integrable systems
  - *Prototypical example: periodic quench/dequench in XXZ*
- Integrability breaking in Richardson-Gaudin
  - *Bethe states as variational wavefunctions*

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