Correlation functions and inner products in Richardson-Gaudin integrability

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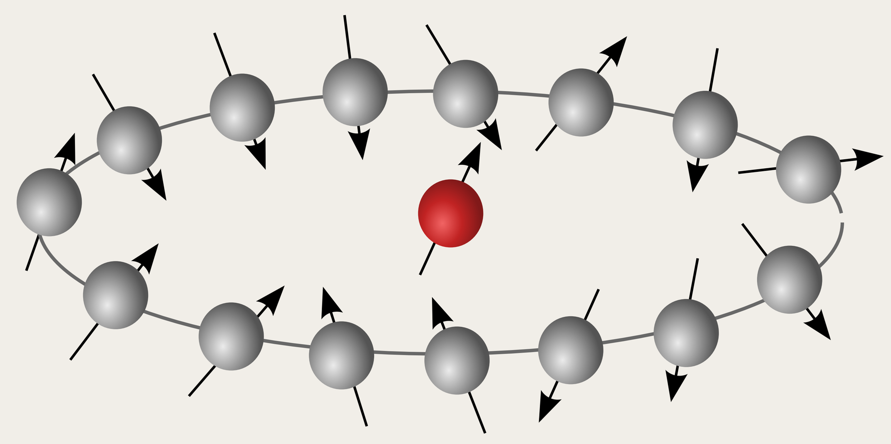
1. Richardson-Gaudin integrable models

- Integrable spin models with **long-range interactions** and **conserved charges**

$$\hat{Q}_i = \left(S_i^z + \frac{1}{2}\right) + g\sum_{j\neq i}^{L} \frac{1}{\epsilon_i - \epsilon_j}\left[\frac{1}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right) + \left(S_i^z S_j^z - \frac{1}{4}\right)\right]$$

- By construction $[\hat{Q}_i, \hat{Q}_j] = 0$
- Richardson-Gaudin Hamiltonians can be constructed as $\hat{H} = \sum \eta_i \hat{Q}_i$
- **Integrability** follows from Algebraic Bethe Ansatz, Generalized Gaudin Algebra,...



E.g. Central spin model

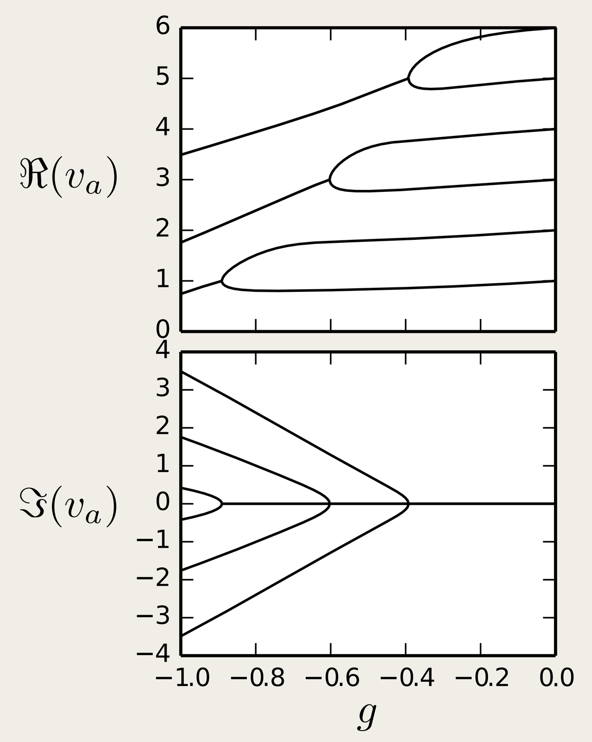
2. Exactly solvable by Bethe ansatz...

- Eigenstates with known product structure
- Defined in terms of **rapidities**

$$|v_1 \dots v_N\rangle = \prod_{a=1}^{N} S^+(v_a)\,|{\downarrow}\dots{\downarrow}\rangle$$
$$= \prod_{a=1}^{N}\left(\sum_{i=1}^{L}\frac{S_i^+}{\epsilon_i - v_a}\right)|{\downarrow}\dots{\downarrow}\rangle$$

satisfying nonlinear Bethe equations

$$\frac{1}{g} + \frac{1}{2}\sum_{i=1}^{L}\frac{1}{\epsilon_i - v_a} - \sum_{b\neq a}^{N}\frac{1}{v_b - v_a} = 0$$

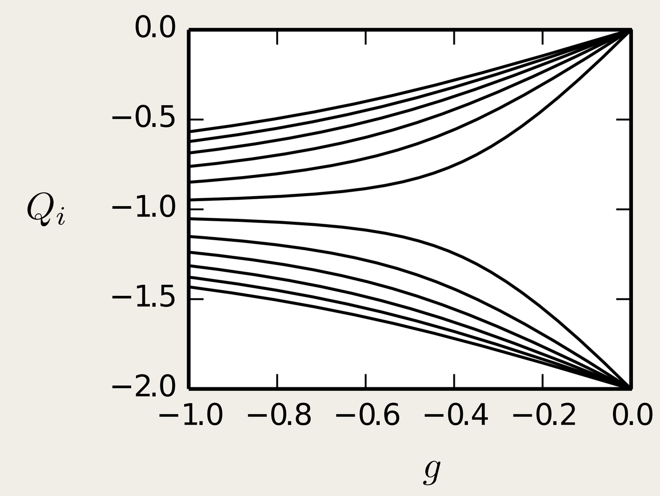


3. ... or through operator identities

- Similar to t-Q framework

$$\hat{Q}_i^2 = \hat{Q}_i - \frac{g}{2}\sum_{j\neq i}^{L}\frac{\hat{Q}_i - \hat{Q}_j}{\epsilon_i - \epsilon_j}$$

- **Quadratic Bethe equations**
- Easier to solve numerically
- Introduced in
  A. Faribault *et al.*, Phys. Rev. B **83**, 235124 (2011)
  A. Faribault and D. Schuricht, J. Phys. A: Math. Theor. **45**, 485202 (2012)



$$\hat{Q}_i\,|v_1 \dots v_N\rangle = Q_i\,|v_1 \dots v_N\rangle = g\sum_{a=1}^{N}\frac{1}{\epsilon_i - v_a}\,|v_1 \dots v_N\rangle$$

4. Inner products can be calculated as determinants

Rapidity-based $\langle v_1 \dots v_N | w_1 \dots w_N\rangle \sim \det K$

$$K_{ab} = \begin{cases} \frac{2}{g} + \sum_{i=1}^{L}\frac{1}{\epsilon_i - x_a} - \sum_{c\neq a}^{2N}\frac{1}{x_c - x_a} & \text{if } a = b \\ -\frac{1}{x_a - x_b} & \text{if } a \neq b \end{cases}$$

with $\{x_a\} = \{v_a\} \cup \{w_b\}$

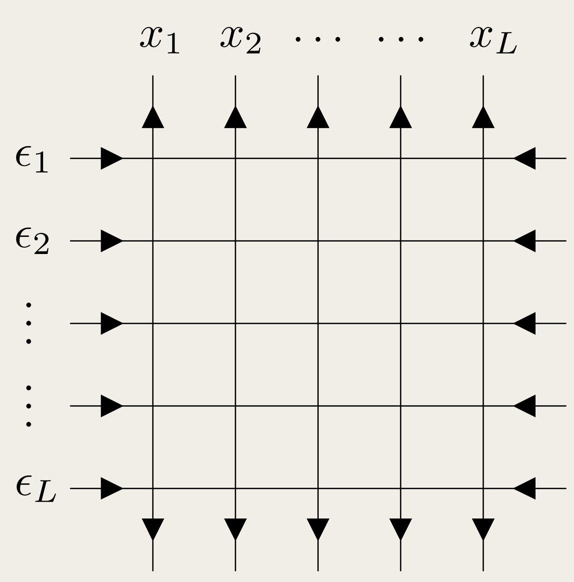
- Can be reduced to Slavnov determinant

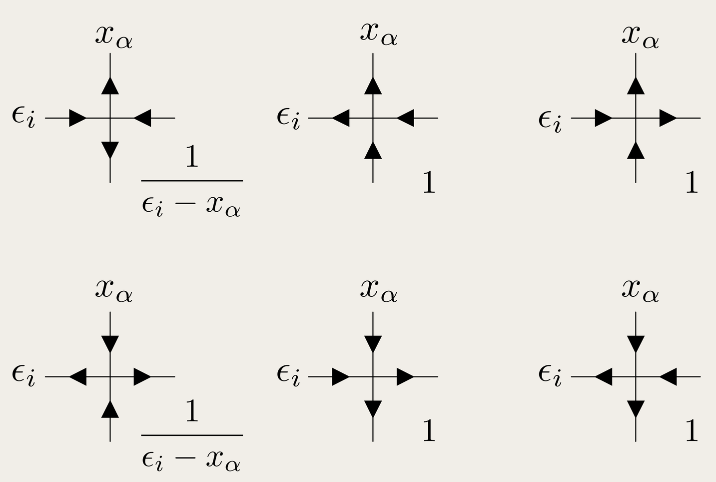
Eigenvalue-based $\langle v_1 \dots v_N | w_1 \dots w_N\rangle \sim \det J$

$$J_{ij} = \begin{cases} \frac{1}{g}\left(2 + Q_i(\{v_a\}) + Q_i(\{w_b\})\right) - \sum_{k\neq i}^{L}\frac{1}{\epsilon_i - \epsilon_k} & \text{if } i = j \\ -\frac{1}{\epsilon_i - \epsilon_j} & \text{if } i \neq j \end{cases}$$

- Does not depend explicitly on rapidities
- Jacobian of Quadratic Bethe equations for normalization

5. Equivalence through DWPFs and Cauchy matrices





- Inner products can be rewritten as **domain wall partition functions**

$$\langle v_1 \dots v_N | w_1 \dots w_N\rangle \sim \langle{\uparrow}\dots{\uparrow}|\prod_{\alpha=1}^{L} S^+(x_\alpha)\,|{\downarrow}\dots{\downarrow}\rangle$$

- DWPF equals **permanent** of Cauchy matrix

$$\mathrm{per}\, C = \frac{\det C * C}{\det C} \quad \text{with} \quad C_{i\alpha} = \langle{\uparrow}|\,S^+(x_\alpha)\,|{\downarrow}\rangle_i = \frac{1}{\epsilon_i - x_\alpha}$$

- Everything can be expressed in terms of **Cauchy matrices**!

$$K \sim \mathbb{1} + C^{-1}(C*C), \qquad J \sim \mathbb{1} + (C*C)C^{-1}$$

- Connection can be made with Slavnov determinant, Gaudin matrix, Izergin-Borchardt determinant,...

6. Inner products lead to correlation functions

- Operator acting on on-shell state = sum of off-shell states

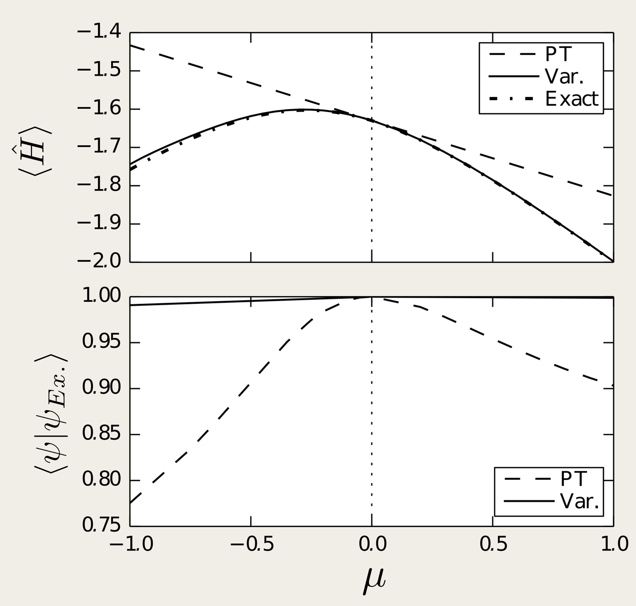
$$S_i^z\,|v_1 \dots v_N\rangle = \sum_a \prod_{b\neq a} S^+(v_b)[S_i^z, S^+(v_a)]\,|{\downarrow}\dots{\downarrow}\rangle + \prod_a S^+(v_a) S_i^z\,|{\downarrow}\dots{\downarrow}\rangle$$

- Correlation function = **sum of determinants**

7. Beyond integrability

- Serves as starting point for
  - **Perturbation theory**
  - Use of Richardson-Gaudin Bethe ansatz as **variational wave function**
- See **arXiv:1707.06793**

$$\hat{H} = \hat{H}_{int} + \mu\hat{V}$$



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