1. Richardson-Gaudin integrable models

- Integrable spin models with long-range interactions and conserved charges
  \[
  \hat{Q}_i = \left( S_i^z + \frac{1}{2} \right) + g \sum_{j \neq i} \frac{1}{\epsilon_i - \epsilon_j} \left[ \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) + \left( S_i^z S_j^z - \frac{1}{4} \right) \right]
  \]
  - By construction \( [\hat{Q}_i, \hat{Q}_j] = 0 \)
  - Richardson-Gaudin Hamiltonians can be constructed as \( \hat{H} = \sum \eta_i \hat{Q}_i \)
  - Integrability follows from Algebraic Bethe Ansatz, Generalized Gaudin Algebra,...

2. Exactly solvable by Bethe ansatz...

- Eigenstates with known product structure
- Defined in terms of rapidities
  \[
  \left| v_1 \ldots v_N \right> = \prod_{i=1}^{N} S^+ \left( \frac{1}{v_i} \right) | \downarrow \ldots \downarrow >
  \]
  satisfying nonlinear Bethe equations
  \[
  \frac{1}{g} + \frac{1}{2} \sum_{i=1}^{N} \frac{1}{\epsilon_i - \epsilon_a} + \sum_{\alpha \neq a} \frac{1}{\epsilon_i - \epsilon_{\alpha}} = 0
  \]

3. ... or through operator identities

- Similar to t-Q framework
  \[
  \hat{Q}_i^2 = \hat{Q}_i - \frac{g}{2} \sum_{j \neq i} \hat{Q}_j
  \]
  - Quadratic Bethe equations
  - Easier to solve numerically
  - Introduced in
  - \( \hat{Q}_i | v_1 \ldots v_N > = Q_i | v_1 \ldots v_N > = g \sum_{a=1}^{N} \frac{1}{\epsilon_i - \epsilon_a} | v_1 \ldots v_N > \)

4. Inner products can be calculated as determinants

- Rapidity-based
  \[
  \langle v_1 \ldots v_N | w_1 \ldots w_N > \sim \det K
  \]
  \[
  K_{ab} = \left( \frac{2 + \sum_{i=1}^{L} x_i x_a - \epsilon_a - \epsilon_b}{-x_a - x_b} \right)
  \]
  with \( \{ x_{a} \} = \{ x_a \} \cup \{ x_b \} \)
  - Can be reduced to Slavnov determinant

- Eigenvalue-based
  \[
  \langle v_1 \ldots v_N | w_1 \ldots w_N > \sim \det J
  \]
  \[
  J_{ij} = \left( \frac{1}{g} \left( 2 + Q_i \{ v_a \} + Q_j \{ w_b \} \right) - \sum_{k \neq i} \frac{1}{\epsilon_i - \epsilon_k} \right)
  \]
  if \( i = j \)
  \[
  - \frac{1}{\epsilon_i - \epsilon_j}
  \]
  if \( i \neq j \)
  - Does not depend explicitly on rapidities
  - Jacobian of Quadratic Bethe equations for normalization

5. Equivalence through DWPFs and Cauchy matrices

- Inner products can be rewritten as domain wall partition functions
  \[
  \langle v_1 \ldots v_N | w_1 \ldots w_N > \sim \langle \{ v_a \} \big| \prod_{a=1}^{N} S^+ (x_a) | \downarrow \ldots \downarrow >
  \]
  - DWPF equals permanent of Cauchy matrix
  \[
  \text{per} C = \frac{\det C + C^*}{\det C}
  \]
  with
  \[
  C_{aa} = \left( \big| \prod_{a=1}^{N} S^+ (x_a) \big| \right)_a = \frac{1}{\epsilon_a - x_a}
  \]
  - Everything can be expressed in terms of Cauchy matrices!
  \[
  K = \mathbb{I} + C^{-1} (C + C^*)
  \]
  \[
  J = \mathbb{I} + (C + C^*) C^{-1}
  \]
  - Connection can be made with Slavnov determinant, Gaudin matrix, Isernin-Borchardt determinant,...

6. Inner products lead to correlation functions

- Operator acting on on-shell state = sum of off-shell states
  \[
  S_i^z | v_1 \ldots v_N > = \sum_{a \neq i \neq \alpha} \prod_b \left( S^+ (v_b) | S_i^z (v_a) \big| \downarrow \ldots \downarrow > \right) + \prod_a \left( S^+ (v_a) | S_i^z (v_a) \big| \downarrow \ldots \downarrow > \right)
  \]
  - Correlation function = sum of determinants

7. Beyond integrability

- Serves as starting point for
  - Perturbation theory
  - Use of Richardson-Gaudin Bethe ansatz as variational wave function
  - See arXiv:1707.06793
  \[
  \hat{H} = \hat{H}_{\text{int}} + \mu \hat{V}
  \]

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