

# Breaking integrability at the boundary

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and



(still!)

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Boundary (sG)

Boundary and bulk ( $\phi^4/\phi^6$ )



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# Plan

1. **Introduction – breaking boundary integrability in sine-Gordon**
2. **First warm-up:  $\phi^4$  kinks and resonant scattering** (old stuff)
3. **Second warm-up:  $\phi^4$  kinks hitting boundaries**
4. **Back to boundary sine-Gordon**
5. **Conclusions**

# 1. Breaking boundary integrability in sine-Gordon

A single classical scalar field  $u(x, t)$  in  $1 + 1$  dimensions, with energy and Lagrangian densities  $\mathcal{E} = \mathcal{T} + \mathcal{V}$  and  $\mathcal{L} = \mathcal{T} - \mathcal{V}$ , where

$$\mathcal{T} = \frac{1}{2}u_t^2 \quad \text{and} \quad \mathcal{V} = \frac{1}{2}u_x^2 + (1 - \cos u),$$

and equation of motion

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

On the full line  $-\infty < x < \infty$  this is (very) well-known to be integrable, with a classical spectrum of kinks, antikinks and breathers.

On the half line  $-\infty < x \leq 0$ , the full two-parameter set of boundary conditions compatible with integrability (and with no additional boundary degrees of freedom) was found by Ghoshal and Zamolodchikov (1994):

$$\left[ u_x + 4K \sin\left(\frac{u - \hat{u}}{2}\right) \right] \Big|_{x=0} = 0,$$

where  $K, \hat{u} \in \mathbb{R}$ .

The GZ integrable boundary conditions:

$$\left[ u_x + 4K \sin \left( \frac{u - \hat{u}}{2} \right) \right] \Big|_{x=0} = 0.$$

Two special cases, (zero) Dirichlet

$$u|_{x=0} = 0$$

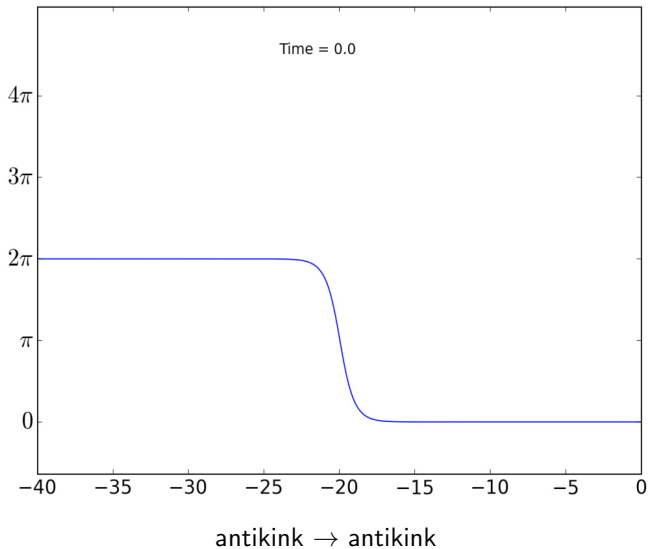
and Neumann

$$u_x|_{x=0} = 0,$$

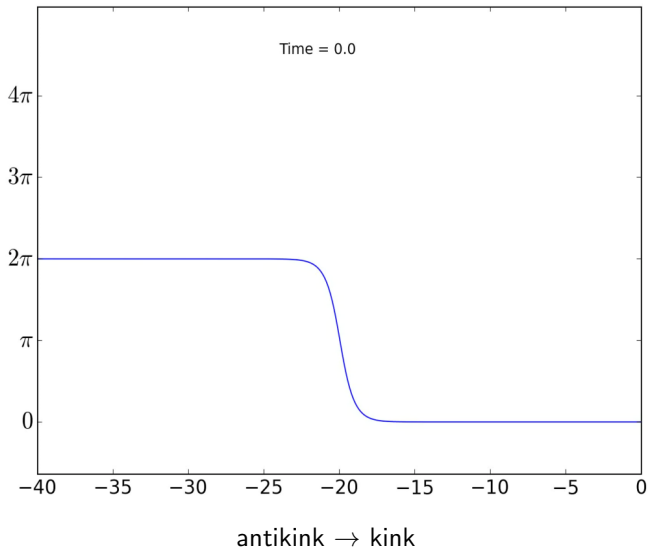
had been known to be integrable before. GZ arrived at their more-general set by a consideration of the lowest-spin extra sine-Gordon conserved charge; soon after, MacIntyre (1995) showed that the full two-parameter family supports an infinite set of conservation laws.

The conservation laws constrain scattering off the boundary to be particularly simple: kinks and antikinks reflect perfectly, as either kinks or antikinks:

Sine-Gordon boundary scattering:  $u|_{x=0} = 0$  (Dirichlet):



Sine-Gordon boundary scattering:  $u_x|_{x=0} = 0$  (Neumann):





... but the real world is not integrable!

So it might be interesting to explore other, non-integrable, boundary conditions - a 'minimal' way to break integrability (just at one point - what harm could that possibly do?).

A natural choice which also interpolates between Dirichlet and Neumann is the (homogeneous) Robin condition.

Instead of the  $\hat{u} = 0$  GZ condition

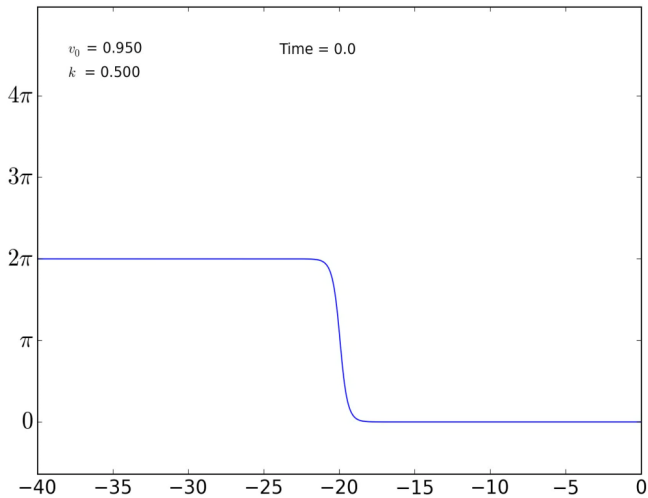
$$\left[ u_x + 4K \sin\left(\frac{u}{2}\right) \right] \Big|_{x=0} = 0,$$

we linearise and impose the Robin condition

$$[u_x + 2ku] \Big|_{x=0} = 0.$$

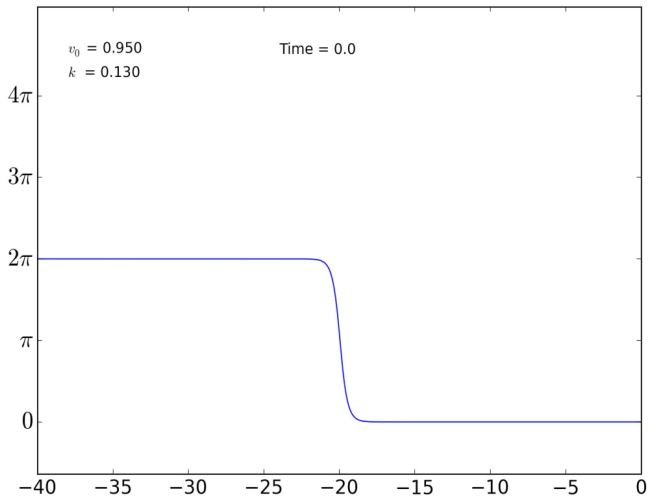
$k \rightarrow \infty$  is Dirichlet;  $k = 0$  is Neumann. Away from these limits, the Robin boundary does not interact nicely with the higher sG conserved charges and boundary scattering becomes much more complicated.

Sine-Gordon boundary scattering (1/5):  $(u_x + u)|_{x=0} = 0$ ,  $v_0 = 0.95$



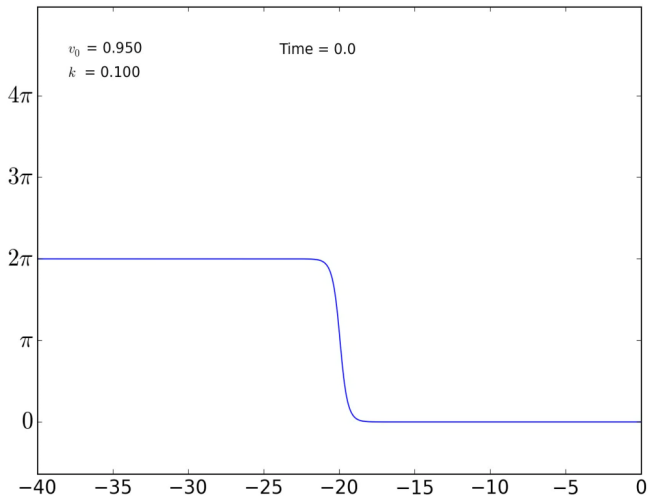
$k = 0.5$  (nearly Dirichlet): antikink  $\rightarrow ?$

Sine-Gordon boundary scattering (2/5):  $(u_x + 0.26 u)|_{x=0} = 0$ ,  $v_0 = 0.95$



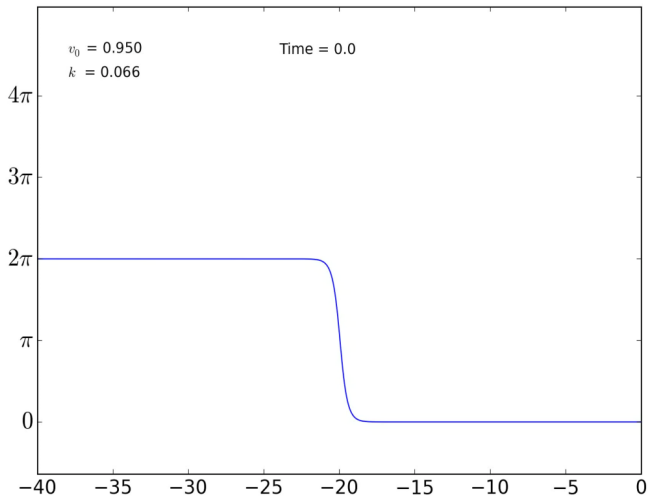
$k = 0.13$ : antikink  $\rightarrow ?$

Sine-Gordon boundary scattering (3/5):  $(u_x + 0.2 u)|_{x=0} = 0$ ,  $v_0 = 0.95$



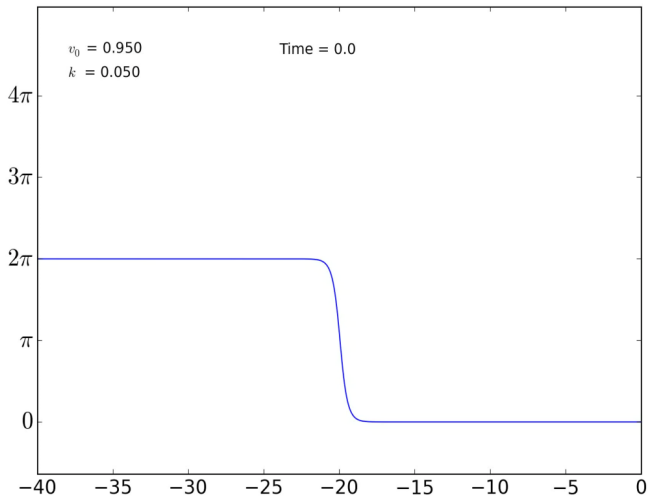
$k = 0.1$ : antikink  $\rightarrow ?$

Sine-Gordon boundary scattering (4/5):  $(u_x + 0.132 u)|_{x=0} = 0$ ,  $v_0 = 0.95$



$k = 0.066$ : antikink  $\rightarrow ?$

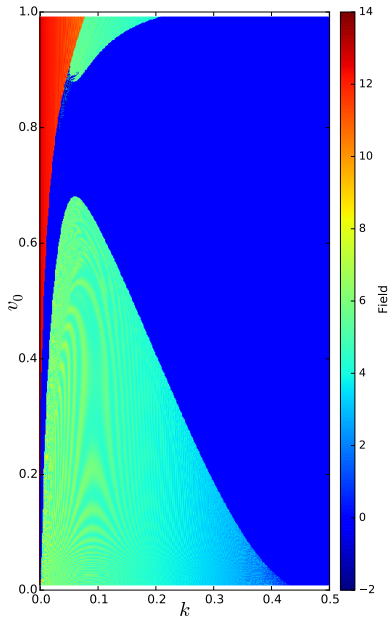
Sine-Gordon boundary scattering (5/5):  $(u_x + 0.1 u)|_{x=0} = 0$ ,  $v_0 = 0.95$



$k = 0.05$  (nearly Neumann): antikink  $\rightarrow ?$

These plots were all for one specific initial velocity,  $v_0 = 0.95$ . What about the overall picture for general  $k$  and  $v_0$ ?

# Robin boundary scattering: phase diagram



A snapshot of  $u_I$ , the late-time field value at  $x = 0$  for the scattering of an initial sine-Gordon antikink with velocity  $v_0$  and on a Robin boundary with parameter  $k$ .

Roughly speaking:

Emitted kink  $\Rightarrow u_I \approx 4\pi$  (red);

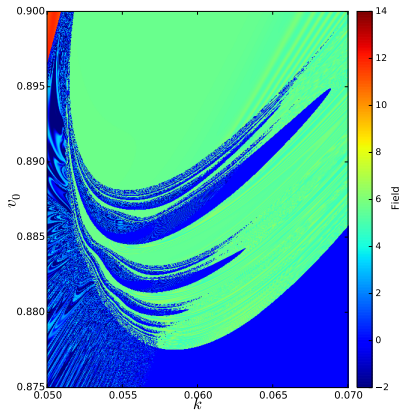
Emitted antikink  $\Rightarrow u_I \approx 0$  (blue);

Neither/both  $\Rightarrow u_I \approx 2\pi$  (light green).

The blur near the top left hides even more complexity. . .



## Robin boundary scattering: zoomed-in phase diagram



A zoomed-in snapshot near the top left of the previous slide.

Dark blue bands correspond to an antikink being emitted; in light green areas breathers, or maybe kink-antikink pairs, are emitted. In between these areas are indeterminate regions where a very slight change in the initial parameters can cause an antikink to be produced or not.

## Questions

- How to disentangle the general final state? What is its soliton content? (A well-defined question, but difficult in practice when breathers are involved!)
- More generally, what's going on? What is the reason for the complicated, almost fractal, structures observed in some parts of the phase diagram?

For the first question, we found that the 'direct' part of the inverse scattering method allowed us to make progress (coming later).

For the second, it turns out that despite being bulk-integrable, the story is particularly complicated for sine-Gordon due to the variety of stable excitations in the bulk theory – not just kinks and antikinks, but also breathers. So I'll illustrate the basic mechanisms first via a simpler example where bulk integrability is also lost: the  $\phi^4$  theory (coming next).

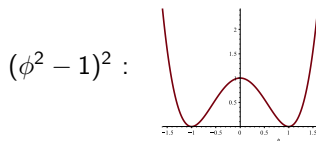
## 2. First warmup: $\phi^4$ kinks and resonant scattering

Switch attention to a scalar field  $\phi(x, t)$  with energy and Lagrangian densities  $\mathcal{E} = \mathcal{T} + \mathcal{V}$  and  $\mathcal{L} = \mathcal{T} - \mathcal{V}$ , where

$$\mathcal{T} = \frac{1}{2}\phi_t^2 \quad \text{and} \quad \mathcal{V} = \frac{1}{2}\phi_x^2 + \frac{1}{2}(\phi^2 - 1)^2 .$$

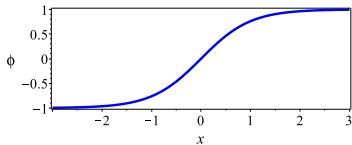
and equation of motion  $\phi_{tt} - \phi_{xx} + 2\phi(\phi^2 - 1) = 0$  (just-about the simplest possible interacting field theory).

- Total energy =  $E[\phi] = \int_{-\infty}^{\infty} \mathcal{E} dx = \frac{1}{2} \int_{-\infty}^{\infty} \phi_t^2 + \phi_x^2 + (\phi^2 - 1)^2 dx$ .
- For  $E[\phi]$  to be finite,  $\phi_t^2$ ,  $\phi_x^2$  and  $(\phi^2 - 1)^2$  must tend to zero as  $x \rightarrow \pm\infty$ .

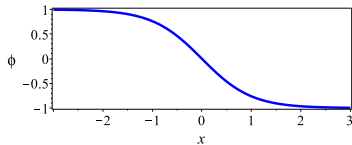


- Hence instead of the infinitely many vacua of sine-Gordon, the theory has just two,  $\phi = \pm 1$ , and finite energy  $\Rightarrow \phi(\pm\infty) \in \{\pm 1\}$ .

- The minimal energy configurations with  $\phi(-\infty) \neq \phi(\infty)$  are called 'topological solitons'; they are the (static) *kinks* and *antikinks*:

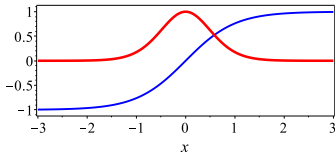


$$\phi_K(x) = \tanh(x - x_0) \quad ,$$

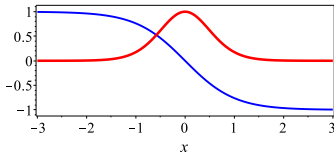


$$\phi_{\bar{K}}(x) = -\tanh(x - x_1)$$

- Their energy densities are localised near to  $x = x_0$  or  $x = x_1$ :



$$\mathcal{E}[\phi_K] = \text{sech}^4(x - x_0) \quad ,$$

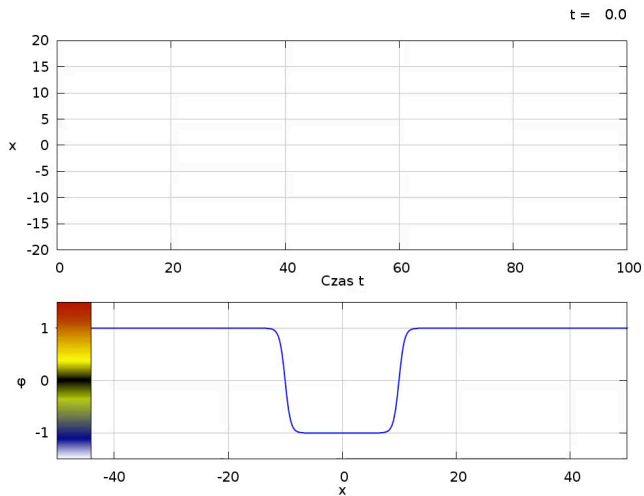


$$\mathcal{E}[\phi_{\bar{K}}] = \text{sech}^4(x - x_1)$$

- The kink and antikink have rest mass  $4/3$ , and *attract* each other with an asymptotic force  $F \sim 32e^{-2R}$ , where  $R = |x_1 - x_0|$ .

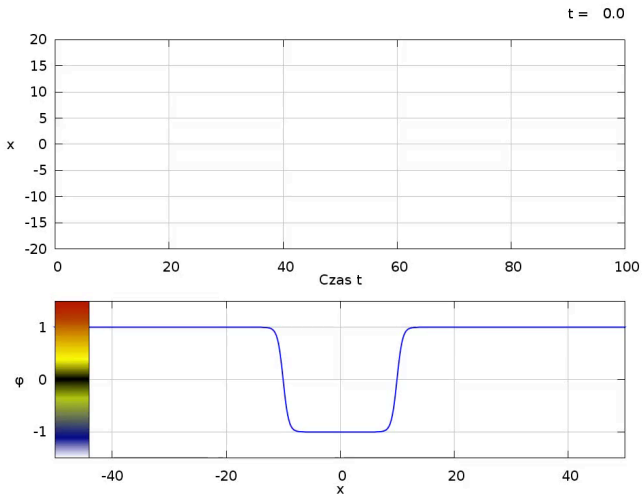
If a  $K$  and  $\bar{K}$  are oppositely-boosted and scattered, then for large enough initial velocities they bounce off each other. However, the theory is not integrable, and so some energy is lost from the translational modes in the process. . .

$\phi^4$  kink scattering:  $v_i = 0.27$



If the initial velocity is reduced below some critical value  $v_c$ , one would expect there to be so little energy left in the translational modes after the collision that the kink and antikink can no longer overcome the attractive force between them and separate, and are instead trapped:

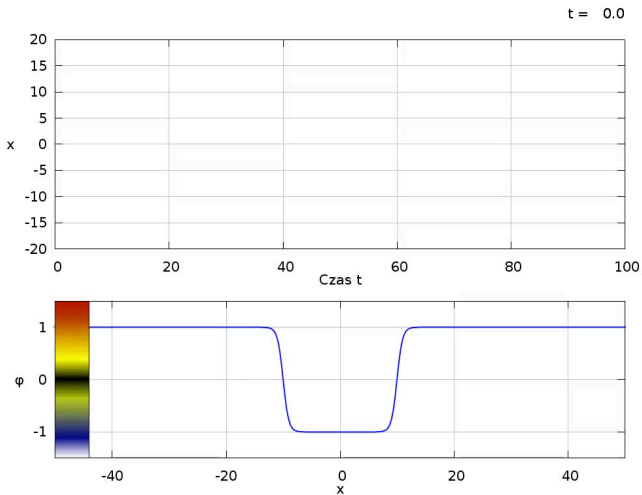
$\phi^4$  kink scattering:  $v_i = 0.27$  ~~0.27~~ 0.24





However there is a surprise waiting if the velocity is reduced further:

$\phi^4$  kink scattering:  $v_i = 0.27 \ 0.24 \ 0.225$

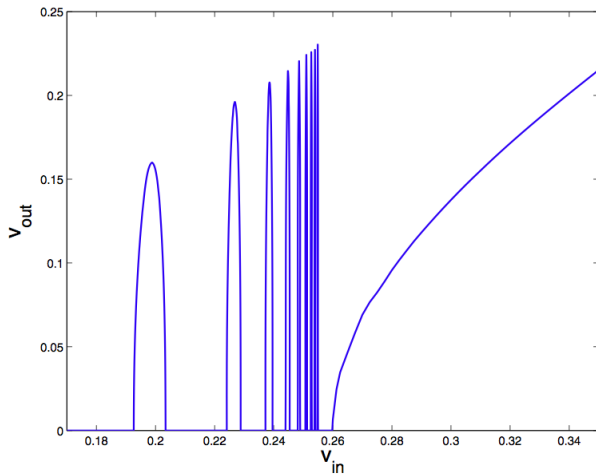


Thus there is at least one 'escape window': a range of velocities below the first trapping velocity  $v_c$  within which the kink and antikink are again able to separate.

This was first observed in the 1970s by, among others, Ablowitz, Kruskal and Ladik. A theoretical explanation was found by Campbell and collaborators in the 1980s and elaborated by many others since; see for example Goodman and Haberman (2005).

The full picture is surprisingly rich. There is an initial sequence of 'two-bounce' windows:

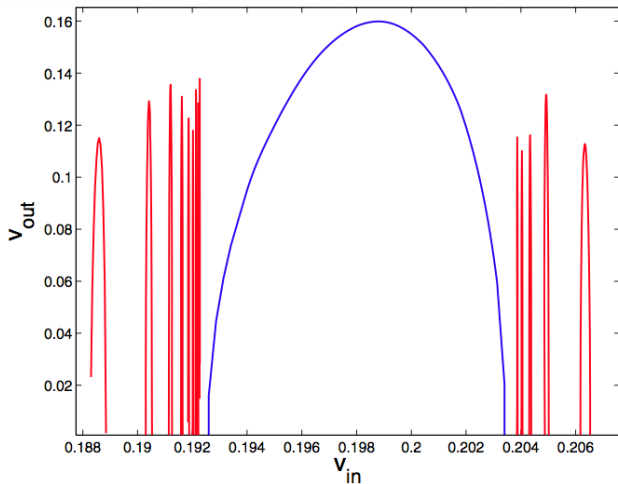
# $\phi^4$ kink scattering: the first windows



(From Goodman and Haberman, 2005)

However at the edges of each of these windows there are sequences of further 'baby windows':

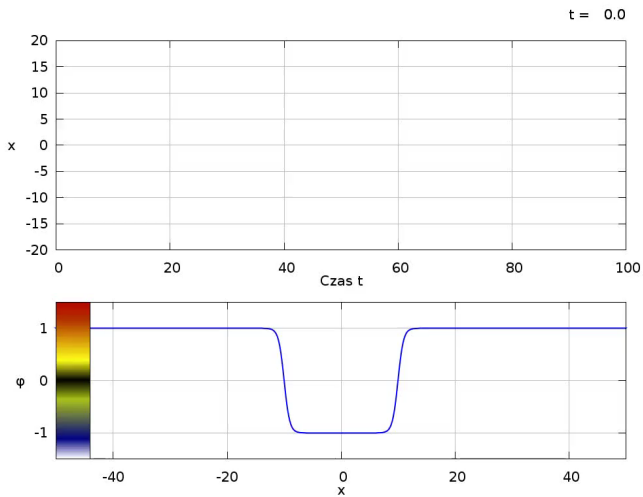
# $\phi^4$ kink scattering: baby windows



(From Goodman and Haberman, 2005)

Inside these windows the kinks bounce *three* times before re-separating:

Inside a three-bounce window:  $v_i = 0.24385$





... and then at the edges of each three-bounce window there are sequences of four-bounce windows, and so on.

## Theoretical treatment

The key point is that the  $\phi^4$  kink has an internal 'wobble' mode.

Take a small oscillation about a single kink  $\phi_K(x) = \tanh(x)$ :

$$\phi(x, t) = \phi_K(x) + \eta(x, t)$$

The e.o.m. for  $\phi$ ,  $\phi_{tt} - \phi_{xx} + 2\phi(\phi^2 - 1) = 0$ , implies for (small)  $\eta$

$$\eta_{tt} - \eta_{xx} + (6\phi_K^2 - 2)\eta = 0$$

or, if  $\eta(x, t) = e^{i\omega t}\chi(x)$ ,

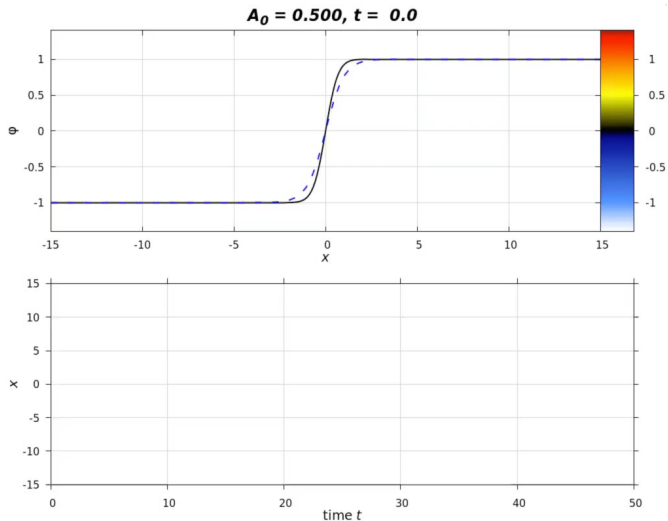
$$-\frac{d^2}{dx^2}\chi - 6\operatorname{sech}^2(x)\chi = (\omega^2 - 4)\chi$$

an eigenvalue problem with *two* eigenvalues ('bound states'),  $\omega = 0, \sqrt{3}$ .

The first is the translational mode; the second is the wobble (absent for sine-Gordon kinks) with period  $2\pi/\sqrt{3} \approx 3.63$ .

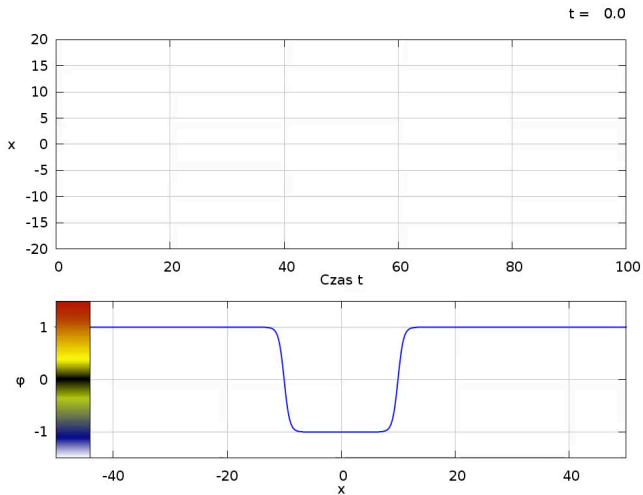
This wobble seen if we start from a distorted kink...

The basic  $\phi^4$  kink wobble:



$$(\phi(x, 0) = \tanh(x) + A_0 \tanh(x)/\cosh(x), \phi_t(x, 0) = 0)$$

... and it is also excited in kink-antikink scattering: ( $v_i = 0.27$  again)



Most of the lost translational energy has been 'parked' in the wobble mode.

For initial velocities  $v$  just below  $v_c$ , the kink and antikink separate after collision but do not quite have the necessary escape velocity to overcome the attractive force between them.

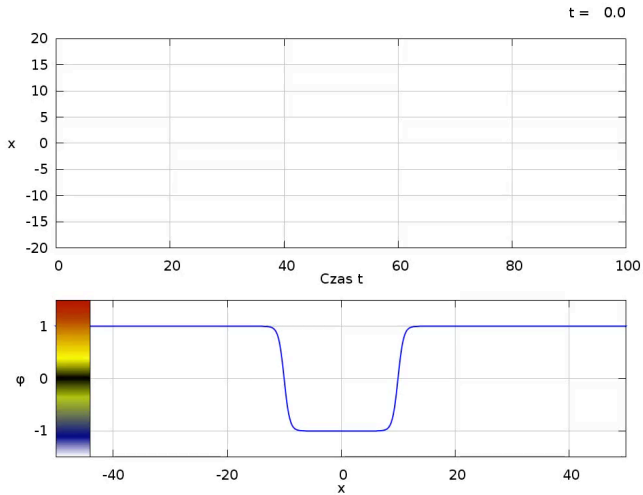
However if on recollision the situation is an approximate **time-reversal** of the initial impact, then the energy stored in the wobbles can be returned to the translational modes and escape is possible at the second attempt.

Note we do **not** need to solve for the nonlinear dynamics of the collision to see that this must work – the argument only uses time-reversal invariance of the equations of motion.

This might happen after one, two or more periods of the internal mode. It might also happen only after two recollisions, or three, and so on, explaining the nested structure of escape windows.

A more quantitative theory can be developed from these ideas but the correctness of the scenario can be seen on re-examining the two-bounce movie. . .

## Two-bounce scattering revisited:



## Key features:

To generate the 'fractal' structures we needed

- ▶ An attractive force putting the kink and antikink at risk of mutual capture;
- ▶ An 'energy storage' mechanism with some periodicity (here, the wobble of the kink) so that this energy could be returned after an integer number of periods, perhaps after multiple recollisions.

This turns out to be rather a general mechanism, observed in many nonintegrable theories. In some cases the energy may be stored *between* the kink and antikink rather than on each one separately (first example:  $\phi^6$  theory). It can also be seen when firing kinks at boundaries. . .

### 3. Second warmup: $\phi^4$ kinks hitting boundaries

Now put the  $\phi^4$  theory on a half line  $-\infty < x < 0$ , with a boundary magnetic field  $H$  placed at  $x = 0$ :

$$L = \int_{-\infty}^0 \left( \frac{1}{2} \phi_t^2 - \frac{1}{2} \phi_x^2 - \frac{1}{2} (\phi^2 - 1)^2 \right) dx + H \phi(0, t)$$

$$\text{Boundary condition: } \phi_x|_{x=0} = H$$

$$\text{Boundary energy: } -H\phi(0, t)$$

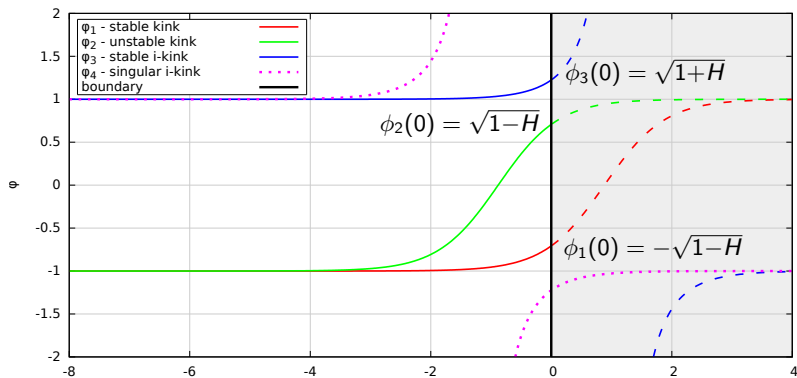
Static kink and antikink solutions on full line are as before:

$$\phi_K(x) = \tanh(x - x_0) \quad , \quad \phi_{\bar{K}}(x) = -\tanh(x - x_0)$$

On a half line, use these to find the ground state by adjusting  $x_0$  so that the boundary condition is satisfied at  $x = 0$ . Note: can also use singular solutions  $\pm \coth(x - x_0)$  so long as the singularity is behind the boundary.



## Static solutions for $0 < H < 1$



$$\phi_1 = \tanh(x-X_0); \quad \phi_2 = \tanh(x+X_0); \quad X_0 = \cosh^{-1}(1/\sqrt{|H|})$$

$$\phi_3 = -\coth(x-X_1); \quad \phi_4 = -\coth(x+X_1); \quad X_1 = \sinh^{-1}(1/\sqrt{|H|})$$

$\phi_3$  is absolutely stable,  $\phi_1$  is metastable, and  $\phi_2$  is unstable.

In fact  $\phi_2$  is a saddle point between  $\phi_1$  and  $\phi_3$ ; at  $H = 1$ ,  $\phi_1$  merges with  $\phi_2$  and the metastable state disappears from the spectrum.

(For  $-1 < H < 0$ , repeat the above with  $\phi \rightarrow -\phi$ .)

## Energies

For these static solutions adapt the Bogomolnyi trick:

$$\begin{aligned} E[\phi] &= \frac{1}{2} \int_{-\infty}^0 ((\phi_x)^2 + (\phi^2 - 1)^2) dx - H\phi|_{x=0} \\ &= \frac{1}{2} \int_{-\infty}^0 (\phi_x \pm (\phi^2 - 1))^2 dx \mp \left[ \frac{1}{3}\phi^3 - \phi \right]_{-\infty}^0 - H\phi|_{x=0} \end{aligned}$$

The integrated term on last line vanishes for kink/antikink profiles and remaining bits rearrange to give

$$E[\phi_2] = \frac{2}{3} + \frac{2}{3}(1-H)^{3/2}$$

$$E[\phi_1] = \frac{2}{3} - \frac{2}{3}(1-H)^{3/2}$$

$$E[\phi_3] = \frac{2}{3} - \frac{2}{3}(1+H)^{3/2}$$

matching the earlier statement that  $\phi_1$  is metastable (a local minimum of the energy),  $\phi_3$  is absolutely stable (the global minimum) and  $\phi_2$  unstable (a saddle point lying between  $\phi_1$  and  $\phi_3$ ).

## Forces and scattering

At  $t = 0$  we fire a single antikink, located at  $x_0 < 0$ , at the boundary, with a velocity  $v_i$ .

For  $H > 0$  the initial boundary profile is  $\phi_1$ , while for  $H < 0$  it is  $-\phi_3$ .

At  $t \approx |x_0|/v_i$  the antikink will hit the wall; but what happens next?

For  $H = 0$  the Neumann boundary condition  $\phi_x|_{x=0} = 0$  can be reflected onto the full line, so an antikink incident on the  $H = 0$  boundary is trapped or reflected from that boundary with exactly the same 'phase diagram' as for full-line kink-antikink scattering.

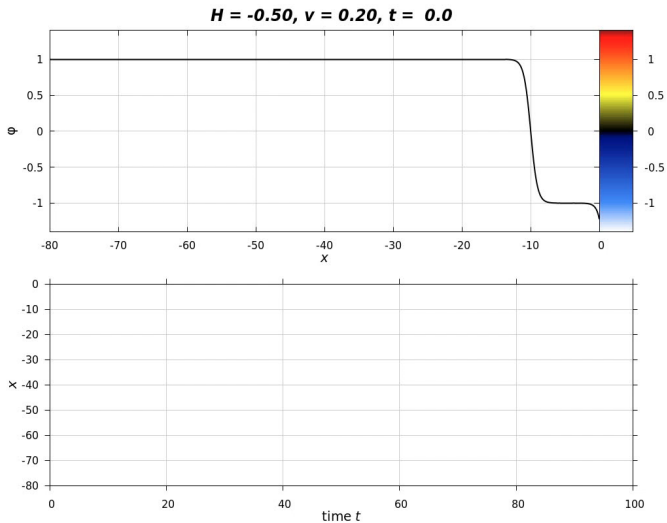
For  $H \neq 0$  the picture distorts. An antikink at  $x_0 < 0$  experiences an asymptotic force

$$F \sim 32 \left( \frac{1}{4}H + e^{2x_0} \right) e^{2x_0}$$

(To calculate  $F$ , use a modified method of images to fit the boundary condition with a full line antikink/(singular) kink solution, and then the bulk force law.)

For  $H < 0$ ,  $F$  is *repulsive* at large distances, *unlike* for the bulk kink and antikink. This means that for small enough initial velocities scattering is almost perfectly elastic as the antikink stays far from the wall. . .

$\phi^4$  boundary scattering:  $H = -0.5$  and  $v_i = 0.2$ :



## Critical velocity for $H < 0$

The initial energy is the sum of the energies of the moving antikink (which has rest mass  $4/3$ ) and the static  $H < 0$  boundary, which is  $-\phi_3$ :

$$E_i(v_i) = \frac{4}{3}(1 - v_i^2)^{-1/2} + \frac{2}{3} - \frac{2}{3}(1-H)^{3/2}$$

The critical velocity  $v_{cr}$  is when the final state is 'at the top of the hill' at the  $-\phi_2$  saddle point, with  $E_i(v_{cr}) = E[-\phi_2]$ :



$$H = -0.4$$

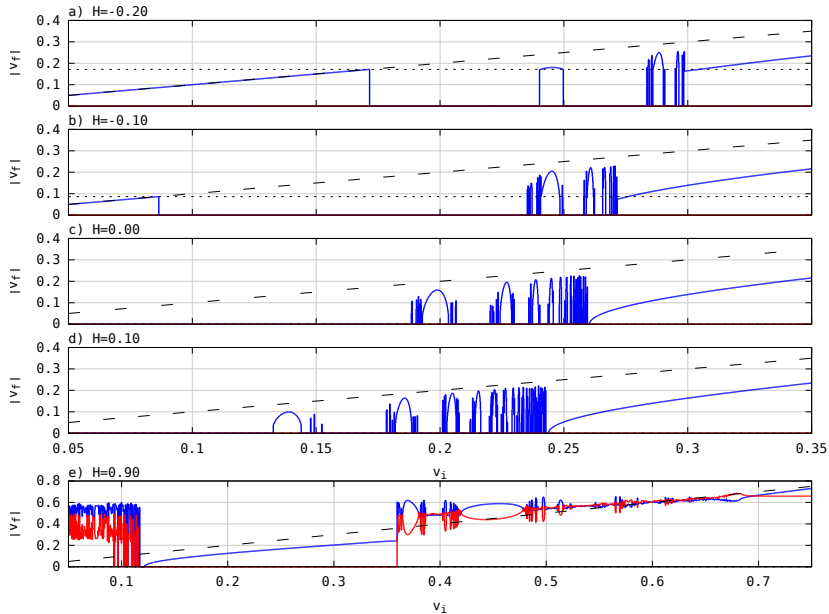
$$v_i = v_{cr}(H) = 0.3332\dots$$

Solving for  $v_{cr}$ ,

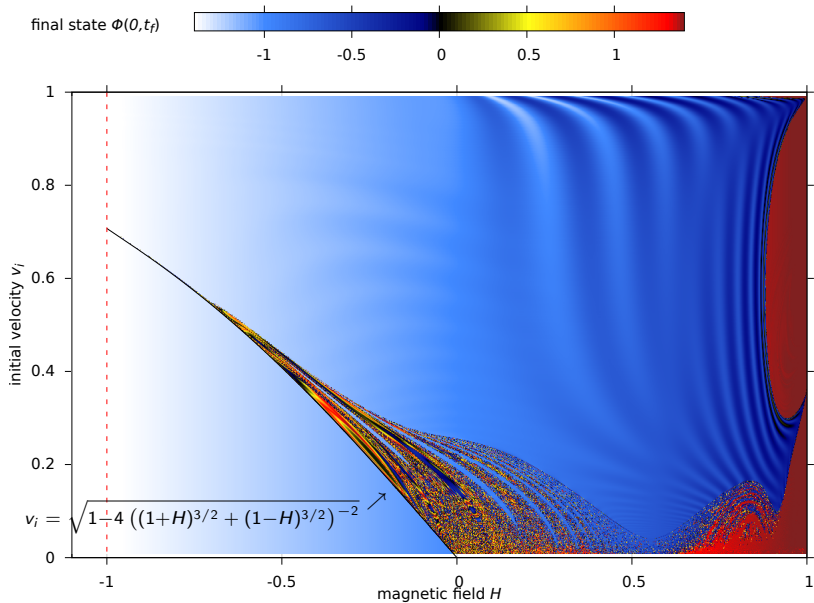
$$v_{cr}(H) = \sqrt{1 - 4 \left( (1+H)^{3/2} + (1-H)^{3/2} \right)^{-2}}$$

For  $v_i > v_{cr}(H)$ , the incoming antikink overcomes the energy barrier, nonlinear effects begin, and life gets complicated again...

# $\phi^4$ boundary scattering: escape windows



# $\phi^4$ boundary scattering: the phase diagram



## Slow-then fast boundary decay

One further feature of the boundary  $\phi^4$  theory: for  $0 < H < 1$  there is an oscillating boundary mode. Its  $H$ -dependent small-amplitude frequency  $\omega_B$  can be found by linearising about the static solution.

For  $H$  small,  $2\omega_B > 2$ , the lowest frequency for bulk radiation, and the second harmonic of the boundary mode can couple to bulk radiative modes, causing it to decay relatively rapidly.

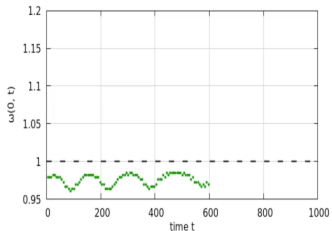
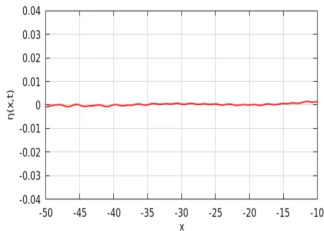
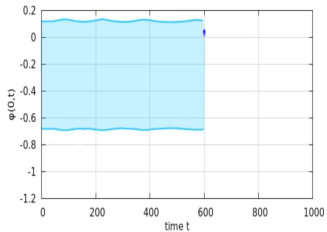
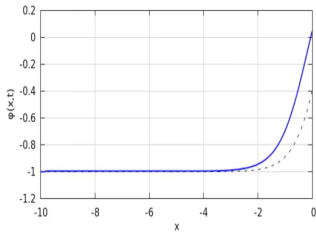
However for larger values of  $H$ ,  $H > H_2 \approx 0.925$ ,  $2\omega_B < 2$  and it is only the *third* harmonic of the boundary oscillation that can couple to the bulk radiation, resulting in a much slower decay rate.

More interestingly, the frequency is also reduced at fixed  $H$  for larger amplitudes of the boundary oscillation, as for the standard pendulum. Suitably tuning  $H$  one can find a situation where a large-amplitude boundary mode has a decay channel forbidden to it, which only opens up once sufficient radiation has been emitted. This 'slow-then-fast' decay is illustrated in the following movie. . .



# $\phi^4$ boundary theory: slow-then-fast decay of the boundary mode

$H = 0.8393, t = 600.055$



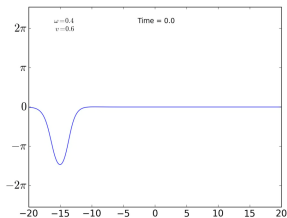
## 4. Back to boundary sine-Gordon

Reminder: in the bulk,

$$u_{tt} - u_{xx} + \sin(u) = 0.$$

Since this model is bulk integrable,

- (a) kinks and antikinks scatter with no loss of velocity; and
- (b) kink-antikink bound states live forever, forming a new class of excitations: the breathers. Here's a moving one:



Now put the model on a half-line,  $x < 0$ , and break integrability by imposing a Robin boundary condition at  $x = 0$ .

The new setup:

$$\begin{aligned}u_{tt} - u_{xx} + \sin(u) &= 0 & (x < 0); \\u_x + 2ku &= 0 & (x = 0).\end{aligned}$$

As before, we fire a kink or antikink at the boundary, and ask about what comes back.

If we wait long enough, all excitations will be far from the boundary, where integrability still holds. There is some sort of 'asymptotic integrability' at work, whereby integrability is only broken for a finite amount of time. This makes the model more interesting to study, but also adds greatly to the possible complexity of the final state, which might contain not only kinks and antikinks but also breathers.

But, it would be tedious to wait long enough for all the solitons and breathers to separate out. Fortunately we don't need to – once everything is far from the boundary (but still tangled up) we can use full-line integrability to extract the kink/antikink/breather content from the numerical data by computer.

## Extracting the soliton content on a full line

... use ideas from inverse scattering theory...

The  $x$  part of the full-line Lax pair is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\frac{i(\phi_x + \phi_t)}{4} & \lambda - \frac{e^{-i\phi}}{16\lambda} \\ \frac{e^{i\phi}}{16\lambda} - \lambda & \frac{i(\phi_x + \phi_t)}{4} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

An eigenfunction decaying at  $x \rightarrow \pm\infty \Rightarrow$  an eigenvalue  $\lambda \in \mathbb{C}$ .

Eigenvalues are either on the positive imaginary axis (kinks or antikinks), or in symmetrically-placed pairs  $(\lambda_n, -\lambda_n^*)$  (breathers).

Their velocities and (in the case of breathers) frequencies are

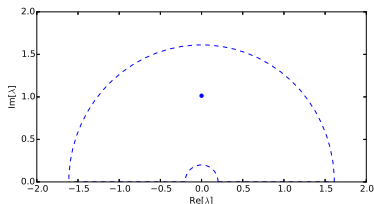
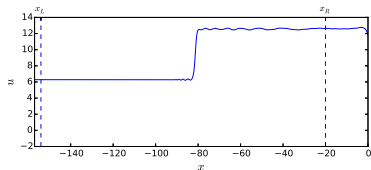
$$v = \frac{1 - 16|\lambda_n|^2}{1 + 16|\lambda_n|^2}, \quad \omega = \frac{\text{Re}[\lambda_n]}{|\lambda_n|},$$

and their energies are

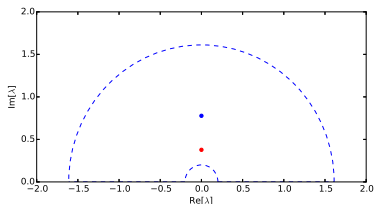
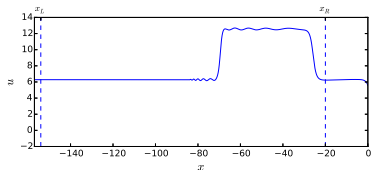
$$E_{\text{soliton}} = \frac{1}{|\lambda_n|} + 16|\lambda_n|, \quad E_{\text{breather}} = 2\text{Im}[\lambda_n] \left( \frac{1}{|\lambda_n|^2} + 16 \right).$$

# Application to the boundary problem

Wait until all excitations have departed from the boundary region, and then patch boundary solution onto full line and compute scattering data to find soliton content of final state:

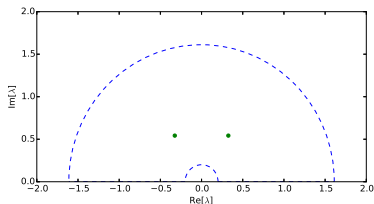
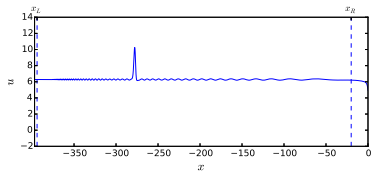


Kink

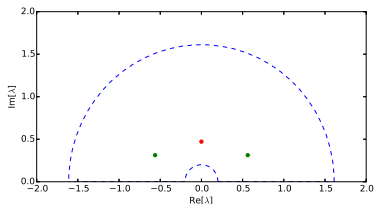
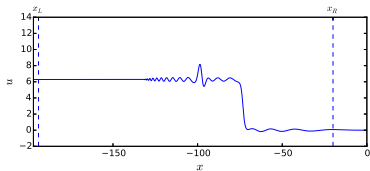


Kink and antikink

# Application to the boundary problem (continued)



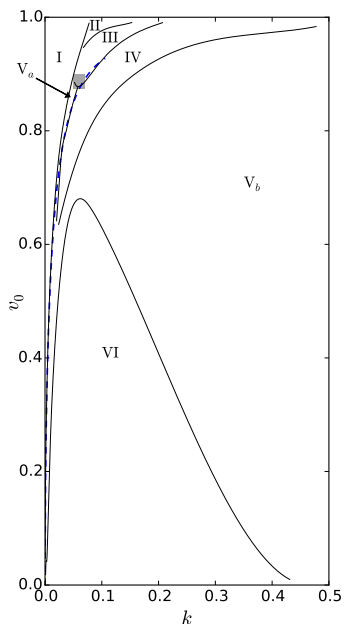
Breather



Antikink and breather

Implement this numerically by searching for zeros of the Wronskian  $W(\lambda) = \text{Det}(\psi_+, \psi_-)$  where  $\psi_{\pm}$  decay as  $x \rightarrow \pm\infty$  to find...

## Robin boundary scattering: final state soliton content



Final states classified by kink, antikink and high energy breather content:

I: Kink

II: Kink and antikink

III: High-energy breather

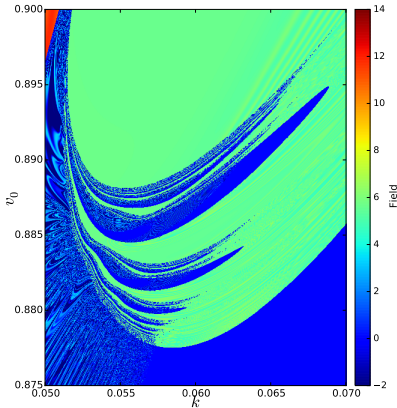
IV: High-energy breather and antikink

V<sub>a</sub> & V<sub>b</sub>: Antikink

VI: None of the above.

Note the match with the earlier snapshot!

## The zoomed-in phase diagram again

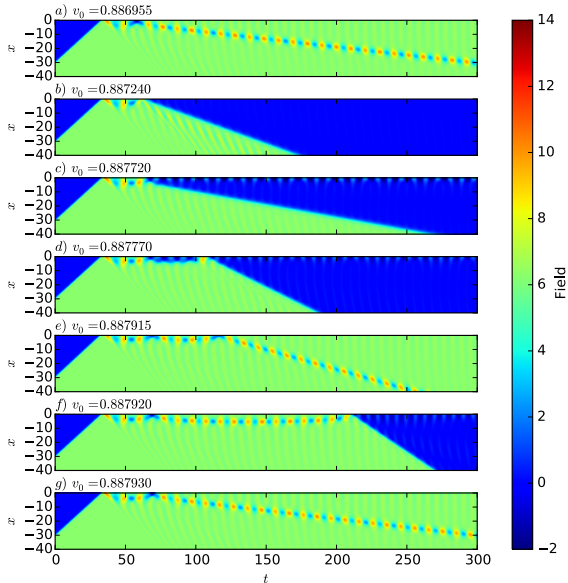


The zoomed-in snapshot shows the late-time values of the field at  $x = 0$  for the shaded area on the previous slide. Dark blue bands correspond to an antikink being emitted; in light green areas only breathers are emitted.

Sections taken at fixed  $k$  exhibit  $v_0$ -dependent windows, similar to those seen in the  $\phi^4$  theory.



# Robin boundary scattering: the resonance mechanism



The key feature behind the 'chaotic' structure: even though the sG kink has no wobble mode, the breather *does* oscillate, and in some regimes it is both produced in the initial boundary collision, and also attracted back to the boundary afterwards. This is enough to get a resonance mechanism to work.

(Plots shown are for  $k = 0.058$ .)

This picture can be backed up by a variety of analytical results, such as calculations of the kink-boundary and breather-boundary forces:

- For an antikink located at  $x_0 < 0$ , park an image kink at  $x_1 > 0$  to form a full-line configuration

$$u(x) = 4 \arctan \left( e^{-(x-x_0)} \right) + 4 \arctan \left( e^{(x-x_1)} \right)$$

For  $|x_0|$  and  $|x_1|$  both large the Robin boundary condition  $(u_x + 2ku)|_{x=0} = 0$  becomes

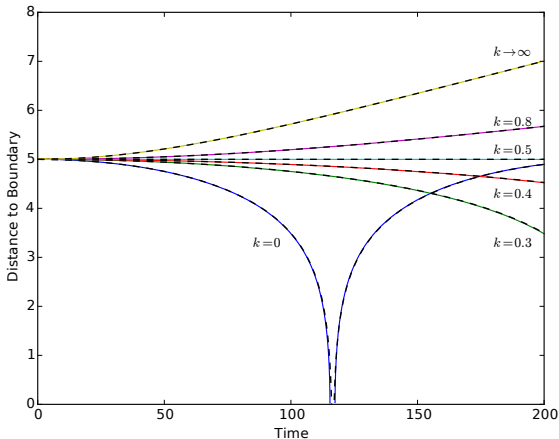
$$4(-e^{x_0} + e^{-x_1}) + 8k(e^{x_0} + e^{-x_1}) = 0.$$

Solving for  $e^{-x_1}$  and computing the force as for  $\phi^4$  yields

$$F = 32 e^{-(x_1-x_0)} = 32 \frac{1-2k}{1+2k} e^{2x_0}.$$

For  $k > 1/2$  an image antikink should be used instead, but the final formula is unchanged, with the force now repulsive instead of attractive.

In the integrable Neumann and Dirichlet limits  $k = 0$  and  $k \rightarrow \infty$  this result matches the asymptotic behaviour of the corresponding exact solutions; it also agrees well at intermediate points, including the 'critical' value  $k_c = 1/2$  at which the predicted force vanishes.



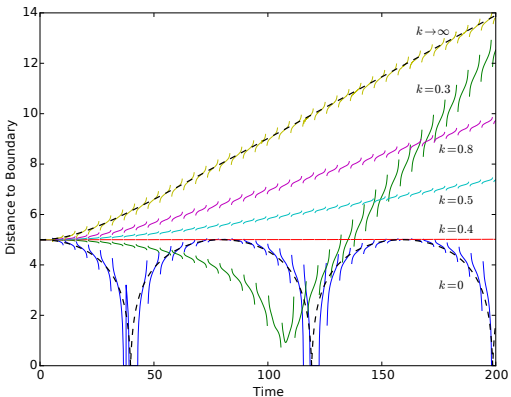
Antikink trajectories near a Robin boundary. (Coloured lines: numerical solutions of the full p.d.e.; dashed lines: predictions from the formula for  $F$ .)

- For breathers the situation is more complicated as we don't have a static solution around which to expand.

The integrable Dirichlet and Neumann limits can be modelled on the full line by adding a symmetrically-placed image breather, exactly in phase with the 'real' breather for the Neumann boundary, and exactly out of phase for Dirichlet.

From the relevant exact two-breather solutions on the full line, it is known that two in-phase breathers feel an attractive force while two out-of-phase breathers experience a repulsive force. Hence a stationary breather is attracted by the  $k = 0$  (Neumann) boundary, while for  $k = \infty$  (Dirichlet) it is repelled.

Numerically we found that the general Robin boundary interpolates between these two limits with a breather-frequency dependent critical velocity at which the force vanishes tending to the value  $k_c = 1/2$  from below as the frequency tends to zero. Recently there's been some analytic progress on this issue (nice idea by Peter Bowcock).



Trajectories of an initially-static breather with frequency 0.6 near to a Robin boundary. (Coloured lines: numerical solutions; Dashed lines: exact trajectories for the Dirichlet (top) and Neumann (bottom) limits.)

These results go most of the way to justifying the claimed resonance mechanism. However a fuller treatment would need some quantitative understanding of the initial bounce, which is still lacking. . .

## 5. Conclusions

- ▶ Classical boundary scattering in sine-Gordon is surprisingly rich once integrability is broken at the boundary.
- ▶ Many features of the 'phase diagram' still to be understood. Would like to develop a more effective collective-coordinate description – difficult as the boundary interaction tends to force the excitation of many other modes, but at least while everything is far from boundary integrability may help.
- ▶ So far integrability was used in a maximally-stupid way, just to disentangle the final state. Is there more that can be done? (Maybe the so-called Fokas method can be applied.)
- ▶ It is also very tempting to ask about the quantum theory, since the space of asymptotic *in* and *out* states should be the same as in the integrable case - this looks to be the ideal half-way-house to a full study of integrability breaking in QFT, where perhaps some of the tools from quantum integrability can be used to study its breakdown.
- ▶ Finally, it would be nice to find an experimental realisation of the resonant scattering phenomenon. AFAIK this is still open!

## Further reading

D.K. Campbell, J.F. Schonfeld & C.A. Wingate,  
*Physica* **9D** (1983) 1 (bulk  $\phi^4$ )

R. Goodman & R. Haberman,  
*SIAM J. Appl. Dyn. Syst.* **4** (2005) 1147 (bulk  $\phi^4$ )

P. Dorey, K. Mersh, T. Romanczukiewicz & Y. Shnir,  
*Phys. Rev. Lett.* **107** (2011) 091602 (bulk  $\phi^6$ )

P. Dorey, A. Halavanau, J. Mercer, T. Romanczukiewicz & Y. Shnir,  
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R. Arthur, P. Dorey & R. Parini,  
*J. Phys. A* **49** (2016) 165205 (boundary sG)



## Journal of Physics A: Mathematical and Theoretical

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### Special issue on fifty years of the Toda lattice

#### Guest Editors

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**Kanehisa Takasaki** Kinki University, Japan

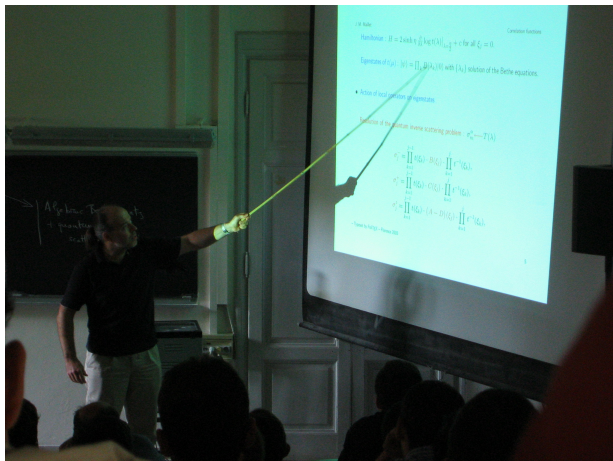
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This is a call for contributions to a special issue of *Journal of Physics A: Mathematical and Theoretical* dedicated to fifty years of the Toda lattice.

<http://iopscience.iop.org/journal/1751-8121/page/Fifty-Years-of-the-Toda-Lattice>



And finally...



Florence, September 2003



Lyon, October 2017

Happy birthday!