Thermal form factors and form factor series for correlation functions of the XXZ chain

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- Includes: From micro- to macro-physics (hierarchy problem)
  1. Derive thermodynamics
  2. Derive long-time large distance asymptotic behaviour of correlation functions
- Approaches
  1. from no interaction to little interaction (perturbation theory)
  2. from one particle to few particles (numerical approaches)
  3. from simple observables to correlation functions (1d integrability)

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- Methods
  1. Representation theory of quantum groups [JMMN 92]
  2. Functional equations of qKZ-type [JM 96, BJMST 04, AK 12]
  3. Algebraic Bethe ansatz based approach [S 89, KMT 99, 00, GKS 04]
  4. Fermionic basis [BJMST 06, 08, JMS 09, BJMS 10]
  5. Algebraic Bethe ansatz based form factor approch [S 89, KMT 99, IKMT 99, KKMST 09, 11A, 11B, 12, DGK 13]
  6. SoV

M-page

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- Invaluable benefit and inspiration

  0: Drinfel'd twists and algebraic Bethe ansatz, 1996

  1: Form factors of the XXZ Heisenberg spin-$\frac{1}{2}$ finite chain, 1999

  2: Correlation functions of the XXZ Heisenberg spin-$\frac{1}{2}$ chain in a magnetic field, 2000

  3: Spin-spin correlation functions of the XXZ-$\frac{1}{2}$ Heisenberg chain in a magnetic field, 2002

  4: Spontaneous magnetization of the XXZ Heisenberg spin-$\frac{1}{2}$ chain, 1999

  5: A form factor approach to the asymptotic behavior of correlation functions in critical models, 2011

Generalized reduced density matrix

- Integrability of XXZ chain based on the underlying quantum group $U_q(\widehat{\mathfrak{sl}}_2)$
- From this: R-matrix, transfer matrix, quantum transfer matrix, reduced density matrix $\big($rather than $H(L)$, $\mathrm{e}^{-H(L)/T}\big)$

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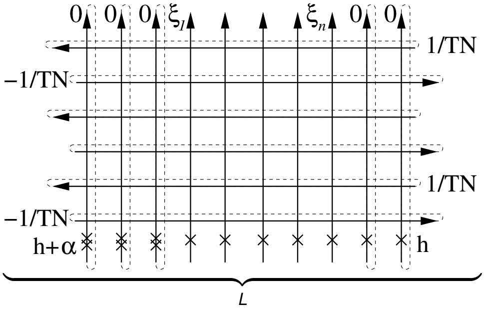

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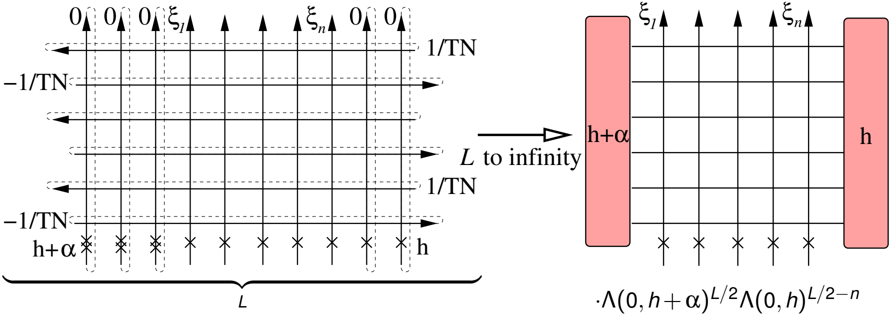

$\cdot\Lambda(0,h+\alpha)^{L/2}\Lambda(0,h)^{L/2-n}$

$$\to D_{[1,n]}(\xi_1,\dots,\xi_n|T,h,\alpha,N) = \frac{\langle h+\alpha|T(\xi_1|h)\otimes\cdots\otimes T(\xi_n|h)|h\rangle}{\langle h+\alpha|\prod_{j=1}^n t(\xi_j|h)|h\rangle}$$

Generalized reduced density matrix

Reduced density matrix and QTM form factor expansion

- Here

$$T(\xi|h) = \mathrm{e}^{\frac{h\sigma^z}{2T}} T(\xi) = \begin{pmatrix} A(\xi|h) & B(\xi|h) \\ C(\xi|h) & D(\xi|h) \end{pmatrix}$$

is the monodromy matrix corresponding to the staggered column-to-column transfer matrix in the picture

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- Using the generalized density matrix we obtain e.g. the transverse two-point functions of the XXZ chain as

$$\begin{aligned}\langle \sigma_1^- \sigma_{m+1}^+ \rangle_N &= \mathrm{Tr}\{ D_{[1,m+1]}(0,\dots,0|T,h,0,N)\, \sigma_1^- \sigma_{m+1}^+ \} \\ &= \frac{\langle \Psi_0 | B(0|h) t(0|h)^{m-1} C(0|h) | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle \Lambda_0(0)^{m+1}} = \sum_\ell A_\ell^{-+} \rho_\ell^m \qquad (*)\end{aligned}$$

  where we have used the notation

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- $(*)$ is a large-distance asymptotic expansion for static correlation functions at finite temperature. Expressions for A_ℓ^{-+} in the Trotter limit $N \to \infty$ were obtained in [M DUGAVE, FG, KK KOZLOWSKI 2013]

Thermal form factor approach

- Calculation of the thermal form factor series consists of three major steps

  Step 1: Analyse the spectral problem of the quantum transfer matrix

  Step 2: Calculate the amplitudes in the Trotter limit

  Step 3: Sum the form factor series

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  Step 3: Sum the form factor series

- We have been working on these problems jointly with MAXIME DUGAVE, KAROL K KOZLOWSKI (since 2012), J SUZUKI (since 2014), M KARBACH and A KLÜMPER (since 2016)

  DGK 2013: J. Stat. Mech. P07010 ✔
  DGK 2014a: SIGMA **10** 043
  DGK 2014b: J. Stat. Mech. P04012
  DGKS 2015a: J. Stat. Mech. P05037
  DGKS 2015b: J. Phys. A **48** 334001 ✔
  DGKS 2016a: J. Phys. A **49** 07LT01
  DGKS 2016b: J. Phys. A **49** 394001 ✔
  GKKKS 2017: arXiv:1708.04062 (JSTAT to appear) ✔

General structure appropriate for taking Trotter limit

- In [DGK 13] we considered $A_n^{\alpha 1}(\xi|\alpha)$ and $A_n^{-+}(\xi|\alpha)$ for finite Trotter number and in the Trotter limit. In both cases the amplitudes consist of three factors

$$A_n^{xy}(\xi|\alpha) = U_{n,s}(\alpha) F_n^{xy}(\xi|\alpha) D_n^{xy}(\alpha)$$

the universal part $U_{n,s}(\alpha)$, the determinant part $D_n^{xy}(\alpha)$ and the factorizing part $F_n^{xy}(\xi|\alpha)$.

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- Conjecture: This structure holds in general, and the factorization of $F_n^{xy}(\xi|\alpha)$ is related to the 'hidden Fermionic structure' of [BOOS, JIMBO, MIWA, SMIRNOV AND TAKEYAMA 2006-10]

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- Derivation based on
  - Scalar product formula [Slavnov 89]
  - NLIE techniques [Klümper 92, 93; G, Klümper, Seel 04]
  - 'Cauchy extraction' [Izergin, Kitanine, Maillet, Terras 99] 
  - Factorization of multiple integrals [Boos, G, Klümper, Suzuki 06; Boos, G 09]

Summation

- In general, at finite temperature, a few terms of the form factor series determine the large-distance asymptotics of the correlation functions. Under certain circumstances, however, we have to sum over infinitely many contributions
  - in the low temperature limit of the static correlation functions
  - if we want the full correlation functions at all distances
  - in the dynamical case

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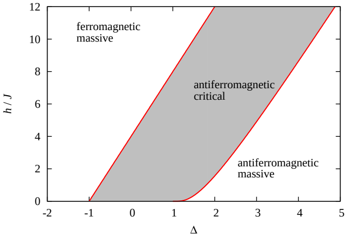
- Recall the ground state phases of the XXZ chain



Static correlations for $T \to 0$

- massless case:
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- massive antiferromagnetic case:
  infinitely many $\xi_n \to \xi_{\max}(h)$ ✔

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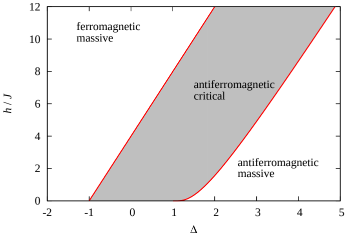

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- Means of summation
  1. Massless case: 'Restricted sum formula' [KKMST 11]
  2. Massive case: auxiliary functions and multiple residue calculus [DGKS 15A, DGKS 16B]

Example large-distance asymptotics for equal times and $T \to 0$

- In [DGKS15A] (using a result of LASHKEVICH 03) we obtained an explicit formula for the next-to-leading term in the asymptotics of the static longitudinal two-point function

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = \frac{(q^2;q^2)_\infty^4}{(-q^2;q^2)_\infty^4}(-1)^m + A \cdot \frac{k(q^2)^m}{m^2}\left((-1)^m - \mathrm{th}^2(\gamma/2)\frac{(q;q^2)_\infty^4}{(-q;q^2)_\infty^4}\right)\left(1+\mathcal{O}(m^{-1})\right)$$

where

$$k(q^2) = \frac{\vartheta_2^2(0|q^2)}{\vartheta_3^2(0|q^2)}, \quad A = \frac{1}{\pi\,\mathrm{sh}^2(\gamma/2)}\frac{(-q;q^2)_\infty^4}{(q^2;q^2)_\infty^2}\frac{(q^4;q^4,q^4)_\infty^8}{(q^2;q^4,q^4)_\infty^8}$$

generalizing the result of the correlation length of JOHNSON, KRINSKY AND MCCOY 73 (recall that $\Delta = (q+q^{-1})/2$, $q = \mathrm{e}^{-\gamma}$)

- Time dependent case can be analyzed in a similar way [DGKS16A]

Asymptotics on phase boundary

- Above asymptotic result holds in the whole antiferromagnetic massive regime $\Delta > 1$, $|h| < h_\ell$, in particular, also if the phase boundary $h = h_\ell$ is approached from below. Hence, to leading order, on the phase boundary

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle \sim (-1)^m \frac{(q^2;q^2)^4}{(-q^2;q^2)^4} \qquad (*)$$

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- Remarkably this can be reproduced if we approach the phase boundary from above and introduce an appropriate scaling function. Using the the techniques developed in [DGK 13a] it can be shown [Dugave 15] that, asymptotically for large m and small positive $h - h_\ell$,

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle \sim (-1)^m \frac{(q^2;q^2)^4}{(-q^2;q^2)^4} \, g(m,h)$$

  where

$$g(m,h) = \frac{\sqrt{\mathrm{e}}\, 2^{1/6}}{A^6} \left(\frac{2k}{1-k^2} \right)^{1/4} \left(\frac{h}{h_\ell} - 1 \right)^{-1/4} \frac{1}{\sqrt{m}}$$

  and A is the Glaisher-Kinkelin constant. Approaching the phase boundary from above in such a way that $g(m,h) = 1$ we reproduce $(*)$

Full form factor series in massive regime for $T \to 0$

- As a result of summing over all excited states and partially turning sums into integrals we obtain the full form factor series e.g. for the longitudinal correlation functions

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = (-1)^m \frac{(q^2;q^2)^4}{(-q^2;q^2)^4}$$

$$+ \sum_{\substack{n \in \mathbb{N} \\ k=0,1}} \frac{(-1)^{km}}{(n!)^2} \int_{-\frac{\pi}{2}-\frac{i\gamma}{2}}^{\frac{\pi}{2}-\frac{i\gamma}{2}} \frac{d^n u}{(2\pi)^n} \int_{-\frac{\pi}{2}+\frac{i\gamma}{2}}^{\frac{\pi}{2}+\frac{i\gamma}{2}} \frac{d^n v}{(2\pi)^n} \, e^{-2\pi i m \sum_{j=1}^n (p(u_j)-p(v_j))}$$

$$\times \mathcal{A}^{zz}(\{u_i\}_{i=1}^n, \{v_j\}_{j=1}^n | k)$$

valid up to multiplicative temperature corrections of the form $\bigl(1 + \mathcal{O}\bigl(T^\infty\bigr)\bigr)$

$$p(x) = \frac{1}{4} + \frac{x}{2\pi} + \frac{1}{2\pi i} \ln\left(\frac{\vartheta_4(x + i\gamma/2, q^2)}{\vartheta_4(x - i\gamma/2, q^2)} \right)$$

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- Series is different from the previously known form factor series which were obtained by the q-vertex operator approach [Jimbo, Miwa 95] or by applying the algebraic Bethe Ansatz approach to the usual transfer matrix [DGKS 15a]

Amplitude densities in longitudinal case

- In the massive parameter regime the amplitudes in the above form factor series were obtained explicitly in [DGKS 16B]

$$\mathcal{A}^{zz}(\{x_i\}_{i=1}^{n_p}, \{y_j\}_{j=1}^{n_p}|k)$$

$$= \left[\frac{2}{(1-q^4)\Gamma_{q^4}\left(\frac{1}{2}\right)G_{q^4}^4\left(\frac{1}{2}\right)}\right]^{2n_p} \left[\prod_{j=1}^{n_p}\left(1-\mathrm{e}^{-2\pi\mathrm{i}F(x_j)}\right)\left(1-\mathrm{e}^{-2\pi\mathrm{i}F(y_j)}\right)\right]$$

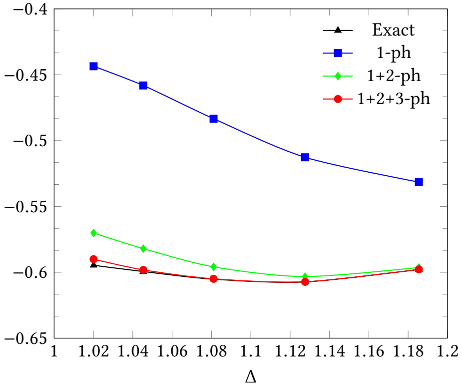
$$\times \left[\prod_{j,k=1}^{n_p} \mathrm{e}^{\varphi(x_j,y_k)-\varphi(y_k,x_j)}\right] \frac{\prod_{1\le j<k\le n_p}\Psi(x_j-x_k)\Psi(y_j-y_k)}{\prod_{j,k=1}^{n_p}\Psi(x_j-y_k)}$$

$$\times \frac{\sin^2\left(\frac{\pi k}{2}+\pi\sum_{j=1}^{n_p}\left(p(y_j)-p(x_j)\right)\right)}{(-q^2;q^2)^4\sin^2\left(\pi F(\theta)\right)} \det_{\mathrm{d}x,[-\pi/2,\pi/2]}(1+\widehat{V}^-) \det_{\mathrm{d}x,[-\pi/2,\pi/2]}(1+\widehat{V}^+)$$

$$\times \det_{m,n=1,\dots,n_p}\left\{\delta_{m,n}+v^-(x_m,x_n)-\int_{-\pi/2}^{\pi/2}\mathrm{d}y\, v^-(x_m,y)R^-(y,x_n)\right\}$$

$$\times \det_{m,n=1,\dots,n_p}\left\{\delta_{m,n}+v^+(y_m,y_n)-\int_{-\pi/2}^{\pi/2}\mathrm{d}y\, R^+(y_m,y)v^+(y,y_n)\right\}$$

Form factor series efficiency



Convergence of the form factor expansion to exact exact value of $g^{zz}(1)$ for various values of Δ. By definition

$$g^{zz}(m) = (-1)^m \langle \sigma_1^z \sigma_{m+1}^z \rangle - \frac{(q^2;q^2)^4}{(-q^2;q^2)^4}$$

which vanishes asymptotically for large m

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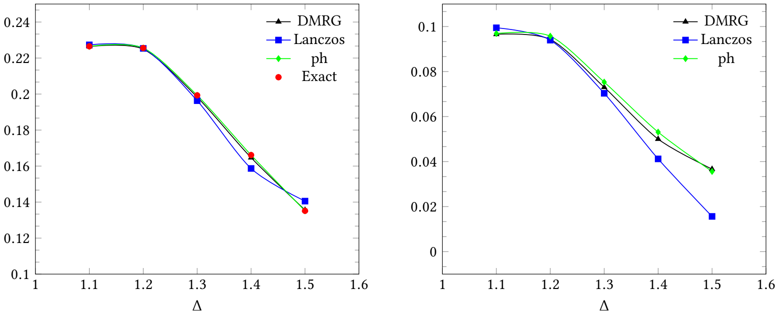
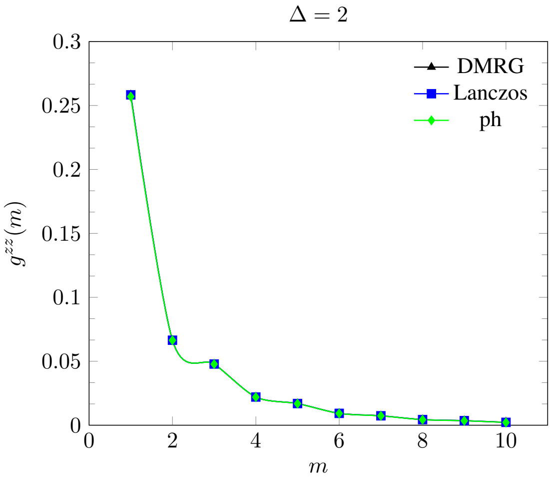


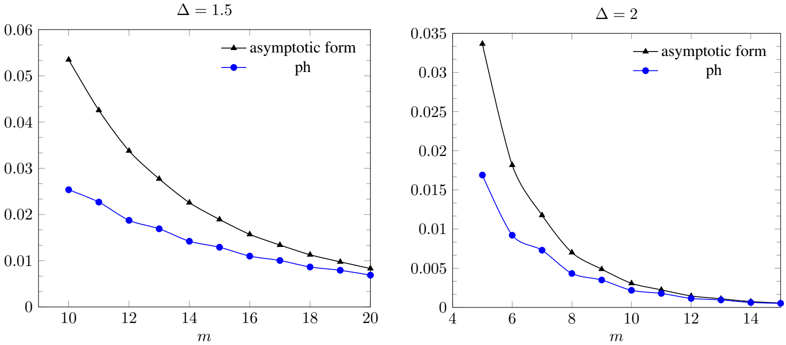
Comparison of $g^{zz}(m)$ estimated by the Lanczos method (squares) and by DMRG (triangles) against $g^{zz}(m)_{\rm ph}$. The spin distance m is 3 (left panel) or 8 (right panel). The red circles in the left panel denote the exact values

Form factor series efficiency



Plots of $g^{zz}(m)$ vs. m for $\Delta = 2$ obtained by three different methods. The curves are almost indistinguishable

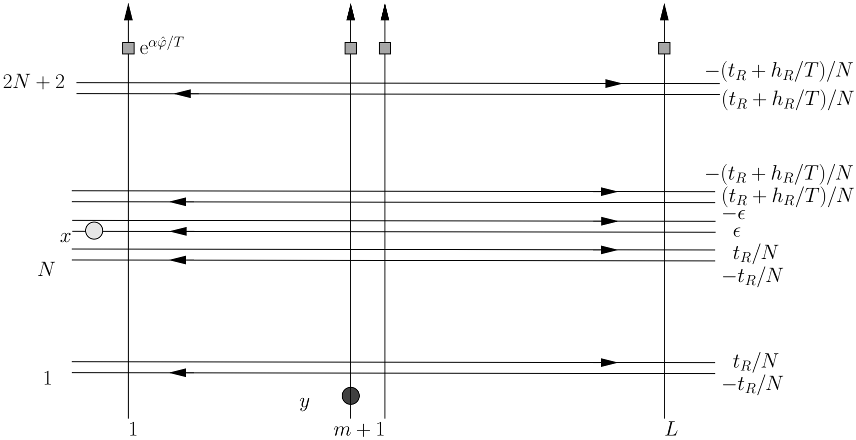
Form factor series vs asymptotic form



Comparison of $g^{zz}(m)$ with its asymptotic form derived in [DGKS 15A]. For $\Delta = 1.5$ (left), due to large ξ, $g^{zz}(m)$ still deviates considerably from its asymptotic form. For $\Delta = 2$ (right) $g^{zz}(m)$ exhibits already a good agreement with the asymptotic form as $\xi \sim 5.29$

Thermal form factor approach to dynamical correlation functions

For the dynamical case we consider the following auxiliary vertex model, normalize by the partition function and take the Trotter limit $N \to \infty$ [K Sakai 2007]



Series representation

THEOREM: The dynamical transverse two-point functions of of the XXZ chain have the form-factor series expansion

$$\big\langle \sigma_1^- \sigma_{m+1}^+(t) \big\rangle_T = \lim_{N\to\infty} \lim_{\varepsilon\to 0} \mathrm{e}^{\mathrm{i}t\alpha s(\sigma^-)} \sum_n \frac{\langle \Psi_0 | B(\varepsilon|\kappa) | \Psi_n \rangle \langle \Psi_n | C(\varepsilon|\kappa) | \Psi_0 \rangle}{\Lambda_n(\varepsilon|\kappa) \langle \Psi_0 | \Psi_0 \rangle \Lambda_0(\varepsilon|\kappa) \langle \Psi_n | \Psi_n \rangle}$$

$$\times \left(\frac{\Lambda_n(0|\kappa)}{\Lambda_0(0|\kappa)} \right)^m \left(\frac{\Lambda_n\big(\frac{t_R}{N}\big|\kappa\big) \Lambda_0\big(-\frac{t_R}{N}\big|\kappa\big)}{\Lambda_0\big(\frac{t_R}{N}\big|\kappa\big) \Lambda_n\big(-\frac{t_R}{N}\big|\kappa\big)} \right)^{\frac{N}{2}}$$

$$= \lim_{N\to\infty} \lim_{\varepsilon\to 0} \mathrm{e}^{-\mathrm{i}ht} \sum_n A_n^{-+}(\varepsilon|\kappa,\kappa) \rho_n^m(0|\kappa,\kappa) \rho_n^{\frac{N}{2}}(t_R/N|\kappa,\kappa) \rho_n^{-\frac{N}{2}}(-t_R/N|\kappa,\kappa)$$

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$$\big\langle \sigma_1^- \sigma_{m+1}^+(t) \big\rangle_T = \lim_{N\to\infty} \lim_{\varepsilon\to 0} \mathrm{e}^{\mathrm{i}t\alpha s(\sigma^-)} \sum_n \frac{\langle \Psi_0 | B(\varepsilon|\kappa) | \Psi_n \rangle \langle \Psi_n | C(\varepsilon|\kappa) | \Psi_0 \rangle}{\Lambda_n(\varepsilon|\kappa) \langle \Psi_0 | \Psi_0 \rangle \Lambda_0(\varepsilon|\kappa) \langle \Psi_n | \Psi_n \rangle}$$

$$\times \left(\frac{\Lambda_n(0|\kappa)}{\Lambda_0(0|\kappa)} \right)^m \left(\frac{\Lambda_n\big(\frac{t_R}{N}\big|\kappa\big) \Lambda_0\big(-\frac{t_R}{N}\big|\kappa\big)}{\Lambda_0\big(\frac{t_R}{N}\big|\kappa\big) \Lambda_n\big(-\frac{t_R}{N}\big|\kappa\big)} \right)^{\frac{N}{2}}$$

$$= \lim_{N\to\infty} \lim_{\varepsilon\to 0} \mathrm{e}^{-\mathrm{i}ht} \sum_n A_n^{-+}(\varepsilon|\kappa,\kappa) \rho_n^m(0|\kappa,\kappa) \rho_n^{\frac{N}{2}}(t_R/N|\kappa,\kappa) \rho_n^{-\frac{N}{2}}(-t_R/N|\kappa,\kappa)$$

Here the amplitudes are of the same form as in the static case [DUGAVE, G, KOZLOWSKI 2013]. All time dependence disappears from the amplitudes in the Trotter limit.

- The sum over n is a sum over all solutions of the Bethe ansatz equations. How to deal with such sums?
- Usual transfer matrix and low-T limit in [Dugave, G, Kozlowski, Suzuki 15, 16]

Partial summation and Trotter limit

- Suggestion: Summation by means of multiple residue calculus and of shell solutions of the non-linear integral equations:

$$\langle \sigma_1^- \sigma_{m+1}^+(t)\rangle_T = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!} \int_{\mathcal{C}_{0,1}} \frac{\mathrm{d}u^n}{(2\pi \mathrm{i})^n} \int_{\overline{\mathcal{C}}_{0,1}} \frac{\mathrm{d}v^{n-1}}{(2\pi \mathrm{i})^{n-1}}$$
$$\times \left[\prod_{j=1}^{n} \frac{\mathrm{e}^{mE(u_j)-t_R \mathrm{e}(u_j)}}{1+\overline{\mathfrak{a}}\big(u_j|\{u\},\{v\},\kappa\big)}\right] \left[\prod_{j=1}^{n-1} \frac{\mathrm{e}^{-mE(v_j)+t_R \mathrm{e}(v_j)}}{1+\mathfrak{a}\big(v_j|\{u\},\{v\},\kappa\big)}\right]$$
$$\times \mathcal{A}^{-+}(0|\{u\},\{v\})\, \mathrm{e}^{-\mathrm{i}ht - \int_{\mathcal{C}_{0,1}} \mathrm{d}\mu\, z(\mu|\{u\},\{v\},\kappa)\big(m\mathrm{e}(\mu) - t_R \mathrm{e}'(\mu)\big)}$$

  for the transverse correlation functions of the XXZ chain

- A similar form factor series representation can be also derived for the longitudinal correlation functions

- Reproduces known results in XX limit

Conclusions

- Finite temperature correlation functions of the XXZ chain can be treated within a thermal form factor approach
- At finite temperature a few terms of the series determine the large-distance asymptotics of the static correlation functions [DGK 13]
- The amplitudes are conjectured to be of the form $A = U \times F \times D$ [DGK 13]
- In the massless regime infinitely many low-lying excitations can be summed up in the low-T limit to obtain the large-distance asymptotics of the two-point functions (CFT + non-CFT amplitudes) to leading non-vanishing order in T [DGK 13, DGK 14b]
- In the massive regime the full thermal form factor series can be written as a series over multiple integrals with explicit integrands [DGKS 15b, DGKS 16b]
- The latter allow us to calculate the two-point functions up to the 3-particle, 3-hole contribution [DGKS 16b] and is numerically very efficient
- Dynamical correlation functions can be treated within the thermal form factor approach. The resulting form-factor series are now being studied

Congratulations!