Entanglement entropies in minimal models from null-vector equations

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in honour of Jean-Michel Maillet

This talk is based on:

Thomas Dupic, Benoît Estienne and YI
Entanglement entropies in minimal models from null vectors
`arXiv:1709.09270`

Outline

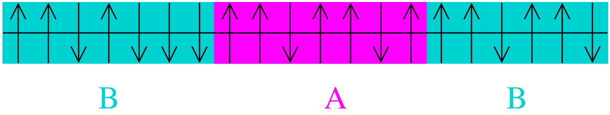
1. Introduction

2. The null-vector approach

3. Entanglement entropies in the Yang-Lee model

1. Introduction

Entanglement entropies in quantum systems



- Density matrix of the whole system $A + B$
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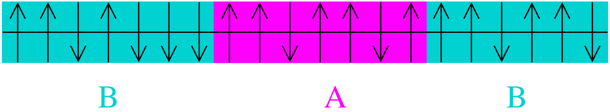

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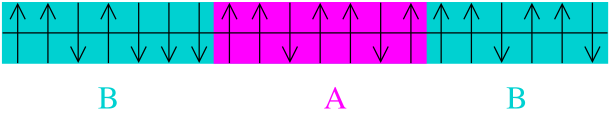

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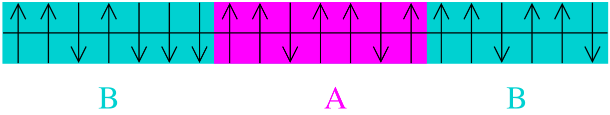

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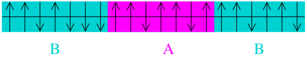
- $\neq$ Gapless systems in $d = 1+1$: **Conformal Field Theory**
  [Holzhey-Larsen-Wilczek '94, Calabrese-Cardy '04]

$$S(A) \sim \frac{c}{3} \log \ell_A$$

The path-integral formalism for Rényi entropies

[Calabrese-Cardy '04]

- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$:



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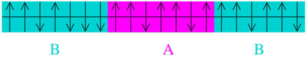

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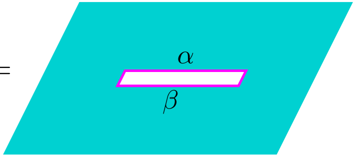

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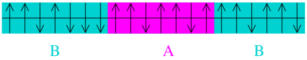

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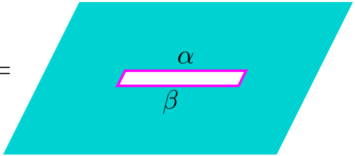
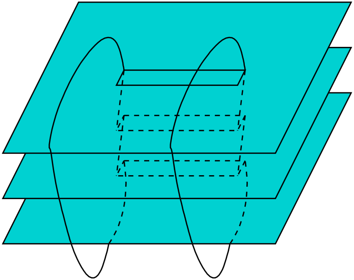


- Rényi entropy

$$\mathrm{Tr}[(\rho_A)^N] =$$



$$= Z(\mathcal{R}_N)$$

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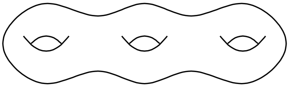
  $\mathcal{R}_N \;\overset{\sim}{=}$ 

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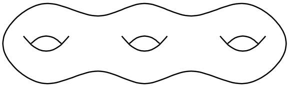
  $\mathcal{R}_N \cong$ 

- For $c = 1$: results available from [Zamolodchikov '87], [Dixon, Friedan, Martinec, Shenker '87], [Alvarez-Gaumé, Gost, Moore, Nelson, Vafa '87], [Dijkgraaf, Verlinde, Verlinde '88]

Overview of Entanglement Entropies in CFT

- Scaling argument $S \propto \frac{c}{3} \log \ell$ [Holzhey, Larsen, Wilczek '94]:
- Path-integral approach, EE by conformal mapping for $A = [u, v]$ [Calabrese,Cardy '04]
- Compute EE at $g > 0$ for $c = 1$ and/or Ising CFTs [Calabrese, Cardy, Tonni, Tagliacozzo, Alba, Misguich, Pasquier, Stéphan, Furukawa, Shiraishi, Essler, Campostrini, Nienhuis '06–'12]
- EE for excited states [Sierra, Alcaraz, Berganza, Palmai '12–'16]
- EE for integrable QFTs [Castro-Alvaredo, Doyon, Cardy, Blondeau-Fournier '07–'15]:
- EE, entanglement spectrum, fidelity using CTM [Franchini, Its, Korepin, Takhtajan, Evangelisti, Weston '11–'12]
- Entanglement after a quench [Cardy '11]
- EE for non-unitary CFTs [Castro-Alvaredo, Doyon, Ravanini, Bianchini, Levi, Couvreur, Jacobsen, Saleur '14–'17]
- Entanglement spectra in FQHE [Li, Haldane, Read, Rezayi, Dubail, Eisler, Peschel, Cardy, Tonni '08–'17]:
- ...

2. The null-vector approach

The $\mathbb{Z}_N$ cyclic orbifold CFT

[Crnković-Sotkov-Stanishkov '89, Klemm-Schmidt '90, Borisov-Halpern-Schweigert '98]

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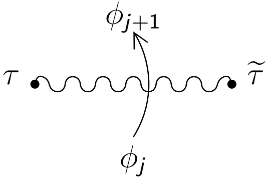

$$\tau(0).(\phi_1, \ldots, \phi_{N-1}, \phi_N)(e^{2i\pi} z) = \tau(0).(\phi_2, \ldots, \phi_N, \phi_1)(z)$$

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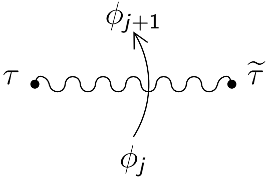

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- Examples:
  - $\langle \tau(u_1)\widetilde{\tau}(v_1) \ldots \tau(u_p)\widetilde{\tau}(v_p) \rangle$
  - $\langle \Phi(\infty)\tau(u)\widetilde{\tau}(v)\Phi(0) \rangle$, $\qquad \Phi := \phi_{12} \otimes \cdots \otimes \phi_{12}$

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Generating algebra [Crnković-Sotkov-Stanishkov '89, Klemm-Schmidt '90, Borisov-Halpern-Schweigert '98]

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$$\left[\widehat{L}_m^{(r)}, \widehat{L}_n^{(s)}\right] = (m-n)\widehat{L}_{m+n}^{(r+s)} + \frac{Nc}{12} m(m^2-1)\, \delta_{m+n,0}\, \delta_{r+s,0}$$

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  - $\widehat{T}^{(r)}(z) \mapsto w^{2-2N} \sum_{j=1}^{N} e^{\frac{2i\pi(j+2)}{N}} T(e^{\frac{2i\pi j}{N}} w) + \frac{c}{24z^2} \left(N - \frac{1}{N} \right) \delta_{r,0}$

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  - $\widehat{L}_m^{(r)} \mapsto \frac{1}{N} L_{Nm} + \frac{c}{24} \left(N - \frac{1}{N} \right) \delta_{r,0} \delta_{m,0}$

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- Basic example: $L_{-1}\mathbf{1} = 0 \quad \Rightarrow \quad \widehat{L}^{(-1)}_{-1/N}\tau = 0$

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From null vectors to differential equations

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- Strategy: obtain linear relation between

$$\langle(\widehat{L}^{(r_1)}_{m_1}\mathcal{O}_1)\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle\,,\langle\mathcal{O}_1(\widehat{L}^{(r_2)}_{m_2}\mathcal{O}_2)\mathcal{O}_3\mathcal{O}_4\rangle\,,$$
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using **orbifold Ward identities** = Closed-contour relations:

$$\oint_C dz\,(z-1)^{m_2+1}\,(z-x)^{m_3+1}\,z^{m_4+1}$$
$$\times\,\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x,\bar{x})\,\widehat{T}^{(r)}(z)\mathcal{O}_4(0)\rangle = 0$$

where $\mathcal{O}_j \in [k_j] \Rightarrow m_j \in \mathbb{Z} + rk_j/N$

Summary of the null-vector approach

1. Consider eigenstate $|\psi\rangle$ of $H_{\mathcal{M}(p,q)}$
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6. Solve consistency for $\{X_i\} \leftrightarrow \{Y_j\}$ $\quad$ ("conformal bootstrap")

3. Entanglement entropies in the Yang-Lee model

The Yang-Lee edge singularity

[Yang-Lee '52]

- Classical Ising model in magnetic field :

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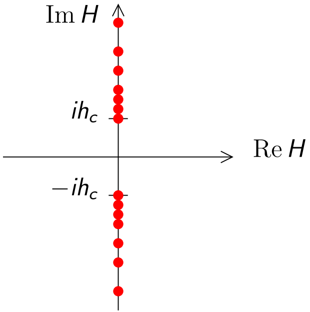

The Yang-Lee model as a minimal model of CFT

[Cardy '85]

- Scaling limit: fixed $T > T_c$ and $H = ih \to ih_c$ [Fisher '78]
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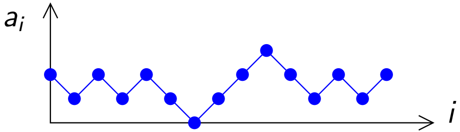
- Exactly solved RSOS model in same universality class [Andrews-Baxter-Forrester '84]

The RSOS quantum chain



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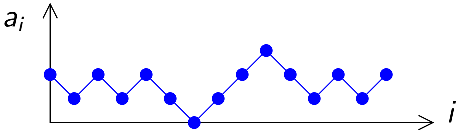

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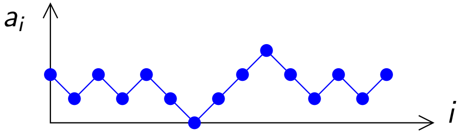

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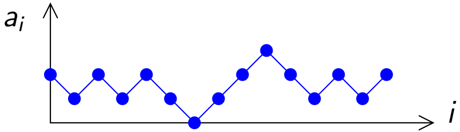

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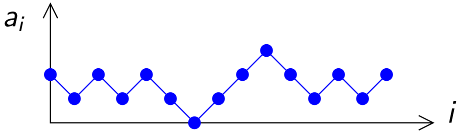

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$$e_i | \dots a_{i-1}, a_i, a_{i+1} \dots \rangle = \delta_{a_{i-1}, a_{i+1}} \sum_{a_i'} \frac{\sin \lambda a_i'}{\sin \lambda a_i} | \dots a_{i-1}, a_i', a_{i+1} \dots \rangle$$

- $\lambda = \frac{\pi(p-q)}{p}$ with $p > q$ coprime $\rightarrow$ scaling limit$=\mathcal{M}(p,q)$
- The YL case : $p = 5, \quad \lambda = \frac{3\pi}{5}$

EEs of a non-unitary model

see also [Bianchini,Castro-Alvaredo,Doyon,Levi,Ravanini '14]

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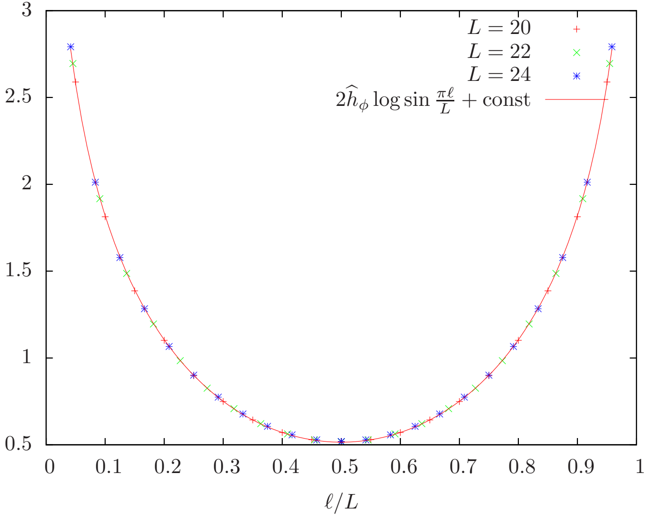
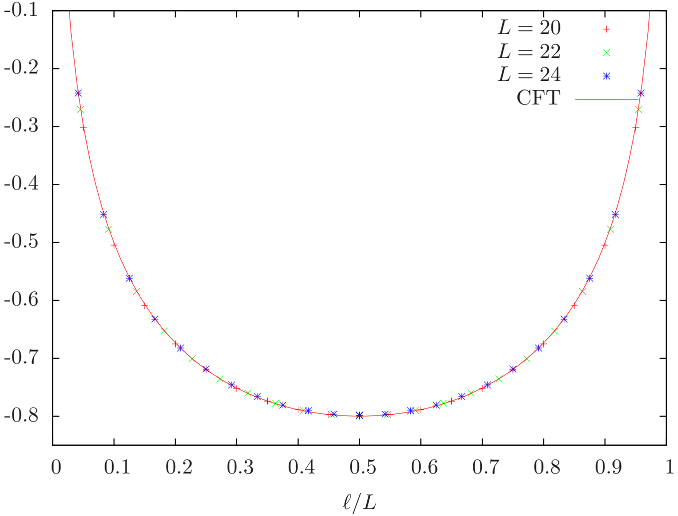
- EE in ground state $|\phi\rangle$:

$$\langle \Phi(\infty)\tau_\phi(1)\widetilde{\tau}_\phi(x,\bar{x})\Phi(0)\rangle = G(x,\bar{x}) \qquad \text{with } \Phi = \phi^{\otimes N}$$

$N = 3$ Rényi entropy in state $|\mathbf{1}\rangle$



$N = 3$ Rényi entropy in the ground state $|\phi\rangle$



Conclusions and perspectives

- Current results
  - "Standard" computations of EE: only $g = 0$ **or** $c \in \{\frac{1}{2}, 1\}$
  - New approach based on $\mathbb{Z}_N$-orbifold of Virasoro algebra
  - Works for minimal models (twist has two null vectors)
  - Applied at $g = 0$ for YL : gives non-trivial $\langle \dots \tau_\phi \widetilde{\tau}_\phi \dots \rangle$
  - Tested at $g = 1$ for YL+Ising : recover $\{\chi_j\}$
- Future work
  - Find systematic derivation of differential equations ?
  - Understand fusion rules in $\mathbb{Z}_N$-orbifold ?
  - Construct Coulomb-Gas formalism for conformal blocks in $\mathbb{Z}_N$-orbifold ? [joint with O. Blondeau-Fournier (Laval)]

Thank you!