Three-point functions in Liouville theory and conformal loop ensembles

Jesper L. Jacobsen [1,2]

[1]Laboratoire de Physique Théorique, École Normale Supérieure, Paris

[2]Université Pierre et Marie Curie, Paris

Correlation functions of quantum integrable systems and beyond, ENS-Lyon, 23 October 2017

Collaborators

Collaborators for this subject:

- Yacine Ikhlef
- Hubert Saleur

More general context includes also four-point functions:

- See Hubert Saleur's talk (after this one)

Computing correlation functions

- Main purpose of statistical physics and statistical field theory.

Computing correlation functions

- Main purpose of statistical physics and statistical field theory.
- Several possible techniques (mainly in two dimensions):
  - Integrability of lattice models.
  - Integrability in the continuum limit.
  - Solvable (conformal) field theories.

Computing correlation functions

- Main purpose of statistical physics and statistical field theory.
- Several possible techniques (mainly in two dimensions):
  - Integrability of lattice models.
  - Integrability in the continuum limit.
  - Solvable (conformal) field theories.
- Several different scopes:
  - Just leading asymptotics (power laws, critical exponents).
  - Refinements (structure constants, scaling corrections, logarithms).
  - Continuum limit, or lattice models (finite separation between points).
  - Local or non-local observables.
  - Bulk or boundary models.
  - One, two, three, four, . . . , N-point functions.

Computing correlation functions

- Main purpose of statistical physics and statistical field theory.
- Several possible techniques (mainly in two dimensions):
  - Integrability of lattice models.
  - Integrability in the continuum limit.
  - Solvable (conformal) field theories.
- Several different scopes:
  - Just leading asymptotics (power laws, critical exponents).
  - Refinements (structure constants, scaling corrections, logarithms).
  - Continuum limit, or lattice models (finite separation between points).
  - Local or non-local observables.
  - Bulk or boundary models.
  - One, two, three, four, ..., N-point functions.

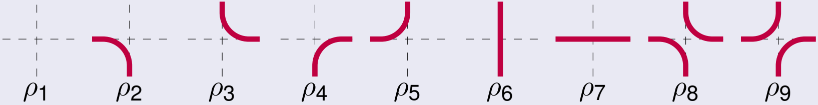
This talk

- Three-point functions (asymptotics including structure constants).
- Bulk theories with non-local observables (clusters and loops).

Coulomb gas and loop models

Loop models in have been extensively studied in two dimensions

Typical example (which is integrable) [Blöte-Nienhuis 1989]:

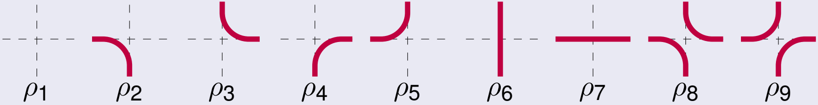


Non-local weight of n per closed loop. Potts model: ρ_8, ρ_9 and $n = \sqrt{Q}$.

Coulomb gas and loop models

Loop models in have been extensively studied in two dimensions

Typical example (which is integrable) [Blöte-Nienhuis 1989]:



Non-local weight of n per closed loop. Potts model: ρ_8, ρ_9 and $n = \sqrt{Q}$.

Coulomb gas approach (1980s)

- Orient each loop independently.
- Set $n = \mathrm{e}^{i\gamma} + \mathrm{e}^{-i\gamma} = 2\cos\gamma$, giving weight $\mathrm{e}^{\pm i\gamma}$ to each orientation.
- Make weights local: $\mathrm{e}^{i\gamma\frac{\alpha}{2\pi}}$ when a loop turns an angle α to the left.
- Oriented loops are level lines of a (compactified) bosonic field ϕ.
- Critical exponents etc can be computed within this field theory.
- Rigorous ($\approx$ equivalent) alternative: SLE_κ, CLE_κ [Schramm,...]

Coulomb gas and $N = 3, 4, \ldots$ correlation functions

- Observables are identified with certain CFT fields ϕ.
- Often $\phi_{r,s} \in$ Kac table, with $r, s \in \mathbb{N}$.

Coulomb gas and $N = 3, 4, \ldots$ correlation functions

- Observables are identified with certain CFT fields ϕ.
- Often $\phi_{r,s} \in$ Kac table, with $r, s \in \mathbb{N}$.

In that case analytical progress is possible

- Indices r, s are interpreted as "charges" [Dotsenko-Fateev].
  - Correlation functions $\neq 0$ only if charge-neutral (after "screening").
  - Integral representation. Monodromy of conformal blocks.
- Moreover, singular state at level rs.
  - Then $\langle \phi_{r,s} \cdots \rangle$ satisfies an ODE (solvable for small rs, e.g. 2)

What about geometrical observables (loops, clusters)?

Boundary case

- $\phi_{1,2}$ inserts a curve (loop = hull of cluster) at the boundary.
- 4-point fcts (4 bdry, or 2 bdry + 1 bulk) satisfy hypergeom. ODE.
- Two nice applications:
  - Proba that percolation cluster connects two arcs of a circle [Cardy].
  - Left-passage probability of SLE_κ curve [Schramm].

What about geometrical observables (loops, clusters)?

Boundary case

- $\phi_{1,2}$ inserts a curve (loop = hull of cluster) at the boundary.
- 4-point fcts (4 bdry, or 2 bdry + 1 bulk) satisfy hypergeom. ODE.
- Two nice applications:
  - Proba that percolation cluster connects two arcs of a circle [Cardy].
  - Left-passage probability of SLE_κ curve [Schramm].

Bulk case: More tricky

- $\phi_{0,1} \times \phi_{0,1}$ marks a bulk point of a loop [Saleur-Duplantier].
- $\phi_{1/2,0} \times \phi_{1/2,0}$ inserts a bulk cluster.
- Challenges: Outside Kac table, and indices $\notin \mathbb{N}$.
  - No differential equations.
  - Bulk fusion of such fields: “in progress” [Gainutdinov, JJ, Saleur].

“Geometrical” version of the Coulomb gas

Basically we have a free bosonic action $\frac{g}{4\pi} \int \mathrm{d}^2 r (\partial \phi)^2$

- When $n \uparrow$, fluctuations in ϕ are smaller, so $g \uparrow$.
- For $n = 2$, Kosterlitz-Thouless transition to non-critical phase.

"Geometrical" version of the Coulomb gas

Basically we have a free bosonic action $\frac{g}{4\pi}\int \mathrm{d}^2 r(\partial\phi)^2$

- When $n\uparrow$, fluctuations in ϕ are smaller, so $g\uparrow$.
- For $n = 2$, Kosterlitz-Thouless transition to non-critical phase.

Background electric charge

- The cylinder geometry is appropriate for transfer matrix setup.
- But winding loops then get a wrong weight $\widetilde{n} = 2$.
- Correct by coupling $\Delta\phi = \phi_{\rm top} - \phi_{\rm bottom}$ to background charge α_0.

"Geometrical" version of the Coulomb gas

Basically we have a free bosonic action $\frac{g}{4\pi} \int \mathrm{d}^2 r (\partial\phi)^2$

- When $n \uparrow$, fluctuations in ϕ are smaller, so $g \uparrow$.
- For $n = 2$, Kosterlitz-Thouless transition to non-critical phase.

Background electric charge

- The cylinder geometry is appropriate for transfer matrix setup.
- But winding loops then get a wrong weight $\widetilde{n} = 2$.
- Correct by coupling $\Delta\phi = \phi_{\rm top} - \phi_{\rm bottom}$ to background charge α_0.

Liouville potential [Kondev 1997]

- Local weight is a periodic functional of ϕ.
- Hence expand on vertex operators $\mathrm{e}^{i\alpha\phi}$.
- Keep only most relevant one, and require its RG marginality.
- This fixes g as a function of n.

Local operators in the CG

Electric (vertex) operators $V_\alpha = e^{i\alpha\phi}$

- $\langle V_\alpha(r_1) V_{-\alpha}(r_2) \rangle$ modifies the weight of loops separating r_1 from r_2.
- We need electric charge neutrality (up to shifts by α_0).

Local operators in the CG

Electric (vertex) operators $V_\alpha = \mathrm{e}^{i\alpha\phi}$

- $\langle V_\alpha(r_1) V_{-\alpha}(r_2) \rangle$ modifies the weight of loops separating r_1 from r_2.
- We need electric charge neutrality (up to shifts by α_0).

Magnetic (vortex) operators $\mathcal{O}_m$

- $\langle \mathcal{O}_m(r_1) \mathcal{O}_{-m}(r_2) \rangle$ makes m defect lines run from r_1 to r_2.
- Permits us to compute the fractal dimension of a loop, etc.

Local operators in the CG

Electric (vertex) operators $V_\alpha = \mathrm{e}^{i\alpha\phi}$

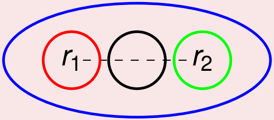
- $\langle V_\alpha(r_1) V_{-\alpha}(r_2) \rangle$ modifies the weight of loops separating r_1 from r_2.
- We need electric charge neutrality (up to shifts by α_0).

Magnetic (vortex) operators $\mathcal{O}_m$

- $\langle \mathcal{O}_m(r_1) \mathcal{O}_{-m}(r_2) \rangle$ makes m defect lines run from r_1 to r_2.
- Permits us to compute the fractal dimension of a loop, etc.

Fundamental conceptual problem in the CG construction

- Many correlators are geometrically well-defined, yet violate charge neutrality. E.g. give four different loop weights like this:



Quantum Liouville theory

- Originates from string theory / quantum gravity [Polyakov 1981].
- Original theory has $c \geq 25$, but recent work [Schomerus, Zamolodchikov, Kostov-Petkova] "tweaks" this to cover $c \leq 1$.

$$\mathcal{A} = \int \mathrm{d}^2 r \frac{\sqrt{g}}{4\pi} \left[\partial_a \phi \partial_b \phi g^{ab} + i\hat{Q}\mathcal{R}\phi + 4\pi\mu \mathrm{e}^{-2i\hat{b}\phi} \right]$$

Quantum Liouville theory

- Originates from string theory / quantum gravity [Polyakov 1981].
- Original theory has $c \geq 25$, but recent work [Schomerus, Zamolodchikov, Kostov-Petkova] "tweaks" this to cover $c \leq 1$.

$$\mathcal{A} = \int \mathrm{d}^2 r \frac{\sqrt{g}}{4\pi} \left[\partial_a \phi \partial_b \phi g^{ab} + i\hat{Q}\mathcal{R}\phi + 4\pi\mu \mathrm{e}^{-2i\hat{b}\phi} \right]$$

- g^{ab} metric; $\mathcal{R}$ Ricci scalar; $\hat{b}$ real constant; and $\hat{Q} = (\hat{b}^{-1} - \hat{b})$.
- The value of μ is irrelevant, as long as $\mu \neq 0$.
- Central charge $c = 1 - 6\hat{Q}^2 \leq 1$.

Quantum Liouville theory

- Originates from string theory / quantum gravity [Polyakov 1981].
- Original theory has $c \geq 25$, but recent work [Schomerus, Zamolodchikov, Kostov-Petkova] "tweaks" this to cover $c \leq 1$.

$$\mathcal{A} = \int \mathrm{d}^2 r \frac{\sqrt{g}}{4\pi} \left[\partial_a \phi \partial_b \phi g^{ab} + i\hat{Q}\mathcal{R}\phi + 4\pi\mu \mathrm{e}^{-2i\hat{b}\phi} \right]$$

- g^{ab} metric; $\mathcal{R}$ Ricci scalar; $\hat{b}$ real constant; and $\hat{Q} = (\hat{b}^{-1} - \hat{b})$.
- The value of μ is irrelevant, as long as $\mu \neq 0$.
- Central charge $c = 1 - 6\hat{Q}^2 \leq 1$.

Vertex operators $V_{\hat{\alpha}} \equiv e^{2\hat{\alpha}\phi}$

- Scaling exponents (conformal weights): $\Delta = \bar{\Delta} = \hat{\alpha}(\hat{\alpha} - \hat{Q})$

Three-point functions and structure constants

DOZZ formula [Dorn-Otto, Zamolodchikov-Zamolodchikov]

$$\left\langle V_{\hat{\alpha}_1}(r_1) V_{\hat{\alpha}_2}(r_2) V_{\hat{\alpha}_3}(r_3) \right\rangle = \hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) \widetilde{\prod} r_{ij}^{-2\Delta_{ij}^k}$$

$$\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \frac{A_{\hat{b}} \Upsilon(\hat{b} - \hat{Q} + \hat{\alpha}_{123}) \widetilde{\prod} \Upsilon(\hat{b} + \hat{\alpha}_{ij}^k)}{\sqrt{\prod_{i=1}^3 \Upsilon(\hat{b} + 2\hat{\alpha}_i) \Upsilon(\hat{b} - \hat{Q} + 2\hat{\alpha}_i)}}$$

Here the $c \leq 1$ version [Schomerus, Kostov-Petkova].

Three-point functions and structure constants

DOZZ formula [Dorn-Otto, Zamolodchikov-Zamolodchikov]

$$\left\langle V_{\hat{\alpha}_1}(r_1) V_{\hat{\alpha}_2}(r_2) V_{\hat{\alpha}_3}(r_3) \right\rangle = \hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) \widetilde{\prod} r_{ij}^{-2\Delta_{ij}^k}$$

$$\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \frac{A_{\hat{b}} \Upsilon(\hat{b} - \hat{Q} + \hat{\alpha}_{123}) \widetilde{\prod} \Upsilon(\hat{b} + \hat{\alpha}_{ij}^k)}{\sqrt{\prod_{i=1}^3 \Upsilon(\hat{b} + 2\hat{\alpha}_i) \Upsilon(\hat{b} - \hat{Q} + 2\hat{\alpha}_i)}}$$

Here the $c \leq 1$ version [Schomerus, Kostov-Petkova].

- $\widetilde{\prod}$ makes (ijk) run over the three cyclic permutations of (123).
- $\hat{\alpha}_{ij}^k \equiv \hat{\alpha}_i + \hat{\alpha}_j - \hat{\alpha}_k$ and $\hat{\alpha}_{123} \equiv \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$.
- $A_{\hat{b}}$ defined by the normalisation $C(\hat{\alpha}, \hat{\alpha}, 0) = 1$.

Three-point functions and structure constants

DOZZ formula [Dorn-Otto, Zamolodchikov-Zamolodchikov]

$$\left\langle V_{\hat{\alpha}_1}(r_1) V_{\hat{\alpha}_2}(r_2) V_{\hat{\alpha}_3}(r_3) \right\rangle = \hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) \widetilde{\prod} r_{ij}^{-2\Delta_{ij}^k}$$

$$\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = \frac{A_{\hat{b}} \Upsilon(\hat{b} - \hat{Q} + \hat{\alpha}_{123}) \widetilde{\prod} \Upsilon(\hat{b} + \hat{\alpha}_{ij}^k)}{\sqrt{\prod_{i=1}^3 \Upsilon(\hat{b} + 2\hat{\alpha}_i) \Upsilon(\hat{b} - \hat{Q} + 2\hat{\alpha}_i)}}$$

Here the $c \leq 1$ version [Schomerus, Kostov-Petkova].

- $\widetilde{\prod}$ makes (ijk) run over the three cyclic permutations of (123).
- $\hat{\alpha}_{ij}^k \equiv \hat{\alpha}_i + \hat{\alpha}_j - \hat{\alpha}_k$ and $\hat{\alpha}_{123} \equiv \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3$.
- $A_{\hat{b}}$ defined by the normalisation $C(\hat{\alpha}, \hat{\alpha}, 0) = 1$.

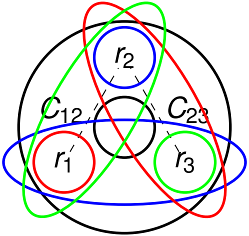
$$\ln \Upsilon(x) \equiv \int_0^\infty \frac{\mathrm{d}t}{t} \left[\left(\frac{\hat{q}}{2} - x \right)^2 \mathrm{e}^{-t} - \frac{\sinh^2 \left(\frac{\hat{q}}{2} - x \right) \frac{t}{2}}{\sinh \frac{\hat{b}t}{2} \sinh \frac{t}{2\hat{b}}} \right]$$

- Defined outside the range $0 < \mathrm{Re}(x) < \hat{q}$ by functional relations.

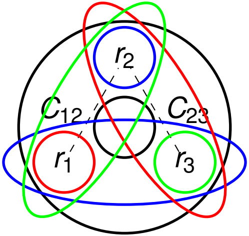
Main results of this talk

- Geometric realisation of DOZZ for loops with modified weights.
- $\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) \neq 0$ even when charge neutrality is broken.
- Our interpretation supports / exploits this lack of charge neutrality.

Definition of our three-point function

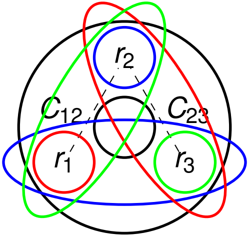


Definition of our three-point function



$(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ linked to loop weights (n_1, n_2, n_3) by matching scaling dimensions of two-point functions: $n_i = 2\cos\pi e_i$ with $\hat{\alpha}_i = \frac{\hat{Q}}{2} + \frac{e_i}{2\hat{b}}$.

Definition of our three-point function



$(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ linked to loop weights (n_1, n_2, n_3) by matching scaling dimensions of two-point functions: $n_i = 2\cos \pi e_i$ with $\hat{\alpha}_i = \frac{\hat{Q}}{2} + \frac{e_i}{2\hat{b}}$.

Convenient reformulation on the cylinder

- Take $r_1 \to -\mathrm{i}\infty$ and $r_3 \to +\mathrm{i}\infty$: No loop can surround r_1 or r_3.
- Loop weight then depends on $[\#\text{traversals mod } 2]$ of C_{12} and C_{23}.

Extracting $\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ from the lattice model

- $Z_{n_i,n_j,n_k}(r_1, r_2, r_3) \equiv Z_{ijk}$ defined by giving modified loop weights
- Set $n = n_0$ for bulk loops.

$$\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3) = Z_{123} \sqrt{Z_{000} \frac{Z_{011}}{Z_{101} Z_{110}} \frac{Z_{202}}{Z_{220} Z_{022}} \frac{Z_{330}}{Z_{033} Z_{303}}}$$

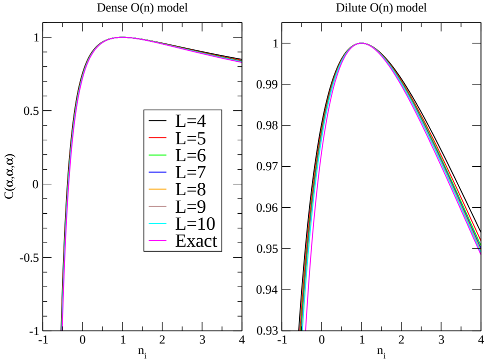
- This is independent of non-universal factors in operator definitions.

Numerical check

- $Z_{n_i,n_j,n_k}(r_1, r_2, r_3)$ obtained from transfer matrix on the cylinder.
- Then form the universal ratio $\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$

Numerical check

- $Z_{n_i,n_j,n_k}(r_1, r_2, r_3)$ obtained from transfer matrix on the cylinder.
- Then form the universal ratio $\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$



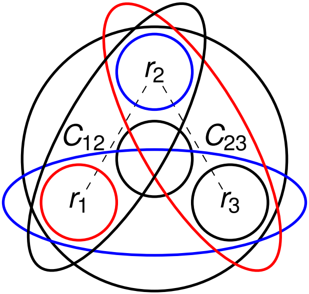
$\hat{C}(\hat{\alpha}, \hat{\alpha}, \hat{\alpha})$ as a function of $n_1 = n_2 = n_3$, in the case $n = 1$.

V_0 is not the identity operator!

- Taking $\hat{\alpha} \to 0$ in $V_{\hat{\alpha}} \equiv e^{2\hat{\alpha}\phi}$ gives of course $\Delta = \bar{\Delta} = 0$.
- Meanwhile $\hat{\alpha}_3 = 0$ implies $(n_1, n_2, n_3) = (n_1, n_2, n)$:

V_0 is not the identity operator!

- Taking $\hat{\alpha} \to 0$ in $V_{\hat{\alpha}} \equiv e^{2\hat{\alpha}\phi}$ gives of course $\Delta = \bar{\Delta} = 0$.
- Meanwhile $\hat{\alpha}_3 = 0$ implies $(n_1, n_2, n_3) = (n_1, n_2, n)$:



- Weight of loop around r_1 depends on whether it *also* surrounds r_3.
- V_0 is an indicator / marking operator (some analogy to SLE_κ).

Two-point functions are orthogonal. . . up to subtleties

- CFT result [Polyakov 1970]: $\langle \Phi(r_1)\Phi(r_2)\rangle = \frac{\delta(\Delta_{\Phi_1},\Delta_{\Phi_2})}{\left|r_1 - r_2\right|^{\Delta_{\Phi_1}+\Delta_{\Phi_2}}}$
- Indeed $\hat{C}(\hat{\alpha},\hat{\alpha},0) = 1$, but $\hat{C}(\hat{\alpha},\hat{\beta},0) \neq 0$ even when $\hat{\alpha} \neq \hat{\beta}$.

Two-point functions are orthogonal. . . up to subtleties

- CFT result [Polyakov 1970]: $\langle \Phi(r_1)\Phi(r_2)\rangle = \frac{\delta(\Delta_{\Phi_1},\Delta_{\Phi_2})}{\left|r_1 - r_2\right|^{\Delta_{\Phi_1}+\Delta_{\Phi_2}}}$
- Indeed $\hat{C}(\hat{\alpha},\hat{\alpha},0) = 1$, but $\hat{C}(\hat{\alpha},\hat{\beta},0) \neq 0$ even when $\hat{\alpha} \neq \hat{\beta}$.

Orthogonality catastrophe [Anderson 1967]

- Consider the overlap $\langle \Omega_2 | \Omega_1 \rangle$ of two XXX ground states with different twists (take $n = 2$).
- $\langle \Omega_2 | \Omega_1 \rangle \propto Z_{n_1,n_2,n_3}(r_1, r_2, r_3)$ with $\hat{\alpha}_3 = \hat{\alpha}_1 + \hat{\alpha}_2$.
- This vanishes as $L^{-\mathrm{const}\times(\alpha_1-\alpha_2)^2}$ with the size L of the chain.

Two-point functions are orthogonal. . . up to subtleties

- CFT result [Polyakov 1970]: $\langle \Phi(r_1)\Phi(r_2)\rangle = \frac{\delta(\Delta_{\Phi_1},\Delta_{\Phi_2})}{\left|r_1 - r_2\right|^{\Delta_{\Phi_1}+\Delta_{\Phi_2}}}$
- Indeed $\hat{C}(\hat{\alpha},\hat{\alpha},0) = 1$, but $\hat{C}(\hat{\alpha},\hat{\beta},0) \neq 0$ even when $\hat{\alpha} \neq \hat{\beta}$.

Orthogonality catastrophe [Anderson 1967]

- Consider the overlap $\langle \Omega_2 | \Omega_1 \rangle$ of two XXX ground states with different twists (take $n = 2$).
- $\langle \Omega_2 | \Omega_1 \rangle \propto Z_{n_1,n_2,n_3}(r_1, r_2, r_3)$ with $\hat{\alpha}_3 = \hat{\alpha}_1 + \hat{\alpha}_2$.
- This vanishes as $L^{-\mathrm{const}\times(\alpha_1-\alpha_2)^2}$ with the size L of the chain.

However this drops out of the universal ratio

The result is $\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_1 + \hat{\alpha}_2) = 1$.

Application to Fortuin-Kasteleyn clusters

- Previously discussed by [Delfino-Viti].
- Numerical cheks in [Picco-Santachiara-Viti-Delfino].

- Take $n_1 = n_2 = n_3 = 0$ and $n = \sqrt{Q}$ for Q-state Potts model.
- Then $\hat{C} \propto \mathbb{P}(r_1, r_2, r_3 \in \text{same FK cluster})$.

Application to Fortuin-Kasteleyn clusters

- Previously discussed by [Delfino-Viti].
- Numerical cheks in [Picco-Santachiara-Viti-Delfino].

- Take $n_1 = n_2 = n_3 = 0$ and $n = \sqrt{Q}$ for Q-state Potts model.
- Then $\hat{C} \propto \mathbb{P}(r_1, r_2, r_3 \in \text{same FK cluster})$.

What about $\mathbb{P}(r_1, r_2, r_3 \in \text{same loop})$?

- We need to change the cluster-inserting operator $\phi_{1/2,0} \times \phi_{1/2,0}$ into the loop-marking operator $\phi_{0,1} \times \phi_{0,1}$.
- But matching scaling dimensions, the DOZZ formula diverges.
- However the numerical measurement is perfectly finite!
- Seemingly DOZZ covers only electric-type operators.