Affine q-deformed symmetry and the classical Yang-Baxter σ-model

François Delduc[1], Takashi Kameyama[1], Marc Magro[1], Benoit Vicedo[2]

1) Laboratoire de Physique, ENS de Lyon, UMR CNRS 5672
2) School of Physics, Astronomy and Mathematics, University of Hertfordshire

Based on: [1701.03691]

Summary

- We show that the local and non-local conserved charges of the YBσM satisfy the defining relations of the infinite-dimensional Poisson algebra $\mathscr{U}_q(L\mathfrak{g})$, which is the classical analogue of the quantum loop algebra $U_q(L\mathfrak{g})$, where $L\mathfrak{g}$ is the loop algebra of $\mathfrak{g}$ (rank(G)>1)

Defining relation of $\mathscr{U}_q(L\mathfrak{g})$

Defining relation of $\mathscr{U}_q(\mathfrak{g})$ [Delduc-Magro-Vicedo, 13]

$$i\{Q^H_{\alpha_i}, Q^H_{\alpha_j}\} = 0,$$
$$i\{Q^H_{\alpha_i}, Q^E_{\pm\alpha_j}\} = \pm A_{ij}\, Q^E_{\pm\alpha_j},$$
$$i\{Q^E_{+\alpha_i}, Q^E_{-\alpha_j}\} = \delta_{ij}\frac{q^{d_i Q^H_{\alpha_i}} - q^{-d_i Q^H_{\alpha_i}}}{q^{d_i} - q^{-d_i}},$$

Simple root: $\alpha_i,\ i = 1, \dots, \mathrm{rank}(G)$ of $\mathfrak{g}$

$$i\{Q^H_{\alpha_i}, \widetilde{Q}^E_{\pm\theta}\} = \pm d_i^{-1}(\theta, \alpha_i)\, \widetilde{Q}^E_{\pm\theta},$$
$$i\{\widetilde{Q}^E_{\theta}, \widetilde{Q}^E_{-\theta}\} = \frac{q^{d_\theta Q^H_\theta} - q^{-d_\theta Q^H_\theta}}{q^{d_\theta} - q^{-d_\theta}},$$
$$i\{Q^E_{\pm\alpha_i}, \widetilde{Q}^E_{\pm\theta}\} = 0,$$

highest root of $\mathfrak{g}$: θ

q-Poisson Serre relations

$$\{\underbrace{Q^E_{\alpha_i}, \{Q^E_{\alpha_i}, \cdots, \{Q^E_{\alpha_i}}_{\mathbf{q}+1\ \text{times}}, \widetilde{Q}^E_{-\theta}\}_q \cdots\}_q\}_q = 0,$$
$$\{\{Q^E_{\alpha_i}, \widetilde{Q}^E_{-\theta}\}_q, \widetilde{Q}^E_{-\theta}\}_q = 0,$$

q is the smallest integer s.t. $-\theta+(\mathbf{q}+1)\alpha_i$ is not a root

- These defining relations are proved without encountering any ambiguity related to the non-ultralocality of the integrable σ-model

Yang-Baxter σ-model

PCM → YB deformation → YBσM

Input: R-matrix satisfying mCYBE

$$S = -\frac{1}{2}(1+\eta^2)^2 \int dt\, dx\, \kappa\left(\partial_+ g g^{-1}, \frac{1}{1-\eta R}\partial_- g g^{-1}\right)$$

Deformation parameter: $\eta \in [0,1)$ [Klimčík, 02,08] [Delduc-Magro-Vicedo, 13]

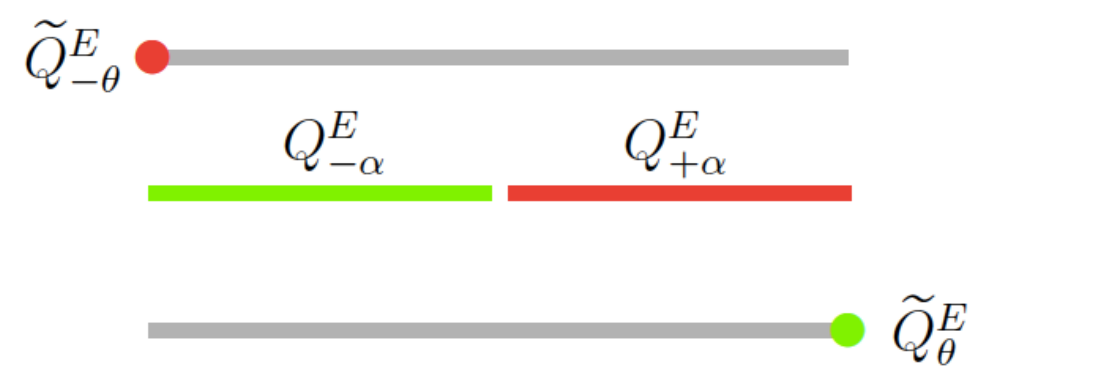
- The YBσM is an integrable deformation of the PCM on a Lie group G
- The deformation breaks the $G\times G$ symmetry to $U(1)^{\mathrm{rank}(G)}\times G$
- R is a skew-symmetric solution of mCYBE

$$\kappa(M, RN) = -\kappa(RM, N), \qquad [RM, RN] = R\big([RM, N] + [M, RN]\big) + [M, N]$$

Here we choose the standard R-matrix of Drinfeld-Jimbo type

- $q \in \mathbb{R}$ is related to η as $q = e^{\gamma}$ with $\gamma = -\eta/(1+\eta^2)^2$

Expansion around the poles of the twist function



- The middle line depicts the level 0 charges of the finite-dimensional $\mathscr{U}_q(\mathfrak{g})$, with the red and green portions corresponding to charges coming respectively from $T^{\mathfrak{g}}(\pm i\eta)$.
- The dots on the ends of the upper and lower lines correspond to two new level ±1 charges of the infinite-dimensional $\mathscr{U}_q(L\mathfrak{g})$, coming respectively from the next order in the expansion of $T^{\mathfrak{g}}(\lambda)$ around $\pm i\eta$.

- The Poisson bracket of the Lax matrix depends on the twist function $\varphi(\lambda)$
- The double pole of $\varphi_{\mathrm{PCM}}(\lambda)$ at $\lambda_0=0$ splits into a pair of single poles of $\varphi_{\mathrm{YB}\sigma\mathrm{M}}(\lambda)$ at $\lambda_\pm=\pm i\eta$
- At the two poles at $\lambda_\pm=\pm i\eta$, $\mathcal{L}^{\mathfrak{g}}(\pm i\eta, x)$ belong to opposite Borel subalgebras of $\mathfrak{g}^{\mathbb{C}}$

⇨ Monodromy matrices $T^{\mathfrak{g}}(\pm i\eta)$ give conserved charges $Q^H_{\alpha_i}$, $Q^E_{\pm\alpha_i}$ which satisfy the defining relations of $\mathscr{U}_q(\mathfrak{g})$ [Delduc-Magro-Vicedo, 13]

- $\widetilde{Q}^E_{\mp\theta}$ associated with the affine simple root are extracted from the linear terms in the expansion of $T^{\mathfrak{g}}(\lambda)$ around $\pm i\eta$, respectively

$$T^{\mathfrak{g}}(i\eta + \epsilon_+) = e^{\gamma \sum_{i=1}^n \int_{-\infty}^{\infty} dx\, h_i(x) H^i}\, P\overleftarrow{\exp}\left[\int_{-\infty}^{\infty} dx \left(\gamma \sum_{\alpha>0} \mathfrak{J}^E_\alpha(x) E^\alpha + \frac{\epsilon_+}{(1+\eta^2)^2}\left[\frac{i}{2}\sum_{\alpha>0}\widetilde{\mathfrak{J}}^E_{-\alpha}(x)E^{-\alpha} + \tilde{\psi}_+\right] + \mathcal{O}(\epsilon_+^2)\right)\right]$$

⇩ Expanding the path-ordered exponential

Charge densities associated with the roots $-\theta + r\,\alpha_i$:

$$\widetilde{\mathfrak{Q}}^E_{\mp\theta}(x) = \widetilde{\mathfrak{J}}^E_{\mp\theta}(x), \qquad \widetilde{\mathfrak{Q}}^E_{-\theta+r\alpha_i}(x) = \widetilde{\mathfrak{J}}^E_{-\theta+r\alpha_i}(x) - \gamma N_{-\theta+(r-1)\alpha_i,\alpha_i}\mathfrak{J}^E_{\alpha_i}(x)\int_{-\infty}^x dy\, \widetilde{\mathfrak{Q}}^E_{-\theta+(r-1)\alpha_i}(y),$$

Defining relations of $\mathscr{U}_q(L\mathfrak{g})$

- Charges associated with the string of roots $-\theta + r\,\alpha_i$ satisfy the q-Poisson brackets, $\{Q^E_{\alpha_i}, \widetilde{Q}^E_{-\theta+r\alpha_i}\}_q = 2iN_{-\theta+r\alpha_i,\alpha_i}\widetilde{Q}^E_{-\theta+(r+1)\alpha_i}$

  Since $-\theta + (\mathbf{q}+1)\,\alpha_i$ is not a root by definition of **q**, we have that $N_{-\theta+\mathbf{q}\alpha_i,\alpha_i} = 0$, hence the q-Poisson-Serre relation is satisfied

- Despite the non-ultralocal nature of the model considered, there are no ambiguities in the Poisson brackets entering the defining relations of the infinite-dimensional Poisson algebra $\mathscr{U}_q(L\mathfrak{g})$
- Unlike the Yangian [MacKay, 92] and SU(2) YBσM [Kawaguchi-Matsumoto-Yoshida, 12], the problematic terms $\partial_x\delta_{xy}$ never showed up in the derivation of the defining relations
- Although the defining relations of $\mathscr{U}_q(L\mathfrak{g})$ are unambiguous, the Poisson brackets of certain conserved charges are still ill-defined

  e.g. $\{\mathfrak{Q}^E_\theta(x), \widetilde{\mathfrak{Q}}^E_{-\theta}(y)\}$ does not appear in the defining relations, but contains $\partial_x\delta_{xy}$ terms