Spectral equations for the modular oscillator

Rinat Kashaev

University of Geneva

Joint work with Sergey Sergeev (Canberra, Australia)

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Motivation: topological strings in toric Calabi–Yau manifolds

To a given (complex three-dimensional) toric CY manifold M, through the mirror symmetry, one can associate a trace class quantum mechanical operator. There is a strong evidence that the spectral properties of that operator are deeply related to enumerative invariants of M encoded into partition functions of topological strings propagating in M. This relation originates from the work of Aganagic–Dijkgraaf–Klemm–Mariño–Vafa (2006) and is expressed most strongly in the form of the conjecture of Grassi–Hatsuda–Mariño (2016).

Example: for the toric CY manifold known as local $\mathbb{P}^1 \times \mathbb{P}^1$ or local $\mathbb{F}_0$, the corresponding operator is of the form

$$\boldsymbol{\rho}_{\mathbb{F}_0,m}^{-1} = \boldsymbol{v} + \boldsymbol{v}^{-1} + \boldsymbol{u} + m\boldsymbol{u}^{-1}, \quad m \in \mathbb{R}_{>0},$$

with positive self-adjoint operators $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfying the Heisenberg–Weyl commutation relation $\boldsymbol{u}\boldsymbol{v} = e^{\mathrm{i}\hbar}\boldsymbol{v}\boldsymbol{u}$, $\hbar \in \mathbb{R}_{>0}$.

Statement of the problem

With normalised Heisenberg operators in the Hilbert space $L^2(\mathbb{R})$ $\boldsymbol{x}\psi(x) = x\psi(x)$ and $\boldsymbol{p}\psi(x) = \frac{1}{2\pi\mathrm{i}}\frac{\partial\psi(x)}{\partial x}$, for $\mathrm{b} \in \mathbb{C}_{\neq 0}$, define

$$\boldsymbol{u} := \mathrm{e}^{2\pi\mathrm{b}\boldsymbol{x}}, \quad \boldsymbol{v} := \mathrm{e}^{2\pi\mathrm{b}\boldsymbol{p}}, \quad \bar{\boldsymbol{u}} := \mathrm{e}^{2\pi\mathrm{b}^{-1}\boldsymbol{x}}, \quad \bar{\boldsymbol{v}} := \mathrm{e}^{2\pi\mathrm{b}^{-1}\boldsymbol{p}}.$$

The common spectral problem for two Hamiltonians $\boldsymbol{H} := \boldsymbol{v} + \boldsymbol{v}^{-1} + \boldsymbol{u} + \boldsymbol{u}^{-1}$ and $\bar{\boldsymbol{H}} := \bar{\boldsymbol{v}} + \bar{\boldsymbol{v}}^{-1} + \bar{\boldsymbol{u}} + \bar{\boldsymbol{u}}^{-1}$ makes sense due to formal commutativity of $\boldsymbol{H}$ and $\bar{\boldsymbol{H}}$, a manifestation of *Faddeev's modular duality*.

In the limit $\mathrm{b} \to 0$ one has $\boldsymbol{H} = 4 + (2\pi\mathrm{b})^2(\boldsymbol{p}^2 + \boldsymbol{x}^2) + \mathcal{O}(\mathrm{b}^4)$.

Pair of functional difference equations:

$\psi(x + \mathrm{i}s) + \psi(x - \mathrm{i}s) = (\varepsilon_s - 2\cosh(2\pi s x))\psi(x)$, $s \in \{\mathrm{b}^{\pm 1}\}$.

Strongly coupled regime $|q| < 1$, $q := e^{\pi\mathrm{i}\mathrm{b}^2}$. The Hamiltonians are Hermitian conjugates of each other if $\mathrm{b} = \mathrm{e}^{\mathrm{i}\theta}$, $0 < \theta < \pi/2$.

In the general case of Baxter's $T - Q$ equations, the solution in the strongly coupled regime is outlined by Sergeev (2005).

The main functional equation

$$f\left(u/q^2\right) + q^2u^2 f(q^2u) = (1 - \varepsilon u + u^2) f(u).$$

Involution in the space of solutions: $f(u) \mapsto \check{f}(u) := u^{-1} f\left(u^{-1}\right)$.
An equivalent first order difference matrix equation

$$\begin{pmatrix} f\left(u/q^2\right) \\ f(u) \end{pmatrix} = L(u) \begin{pmatrix} f(u) \\ f(q^2u) \end{pmatrix}, \; L(u) := \begin{pmatrix} 1 - \varepsilon u + u^2 & -q^2u^2 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} f\left(u/q^2\right) \\ f(u) \end{pmatrix} = M_n(u) \begin{pmatrix} f\left(q^{2n-2}u\right) \\ f(q^{2n}u) \end{pmatrix}, \quad \forall n \in \mathbb{Z}_{>0},$$

$$M_n(u) := L(u)L(q^2u)\cdots L(q^{2(n-1)}u), \quad M_\infty(u) = \begin{pmatrix} \chi_q\left(u/q^2\right) & 0 \\ \chi_q(u) & 0 \end{pmatrix},$$

where $\chi_q(u) = \chi_q(u, \varepsilon)$ is an entire function of $u \in \mathbb{C}$ normalised so that $\chi_q(0) = 1$ and which solves the main functional equation. The second solution $\check{\chi}_q(u) := u^{-1}\chi_q(u^{-1})$ leads to a non-zero Wronskian $[\chi_q, \check{\chi}_q](u) := \chi_q\left(q^{-2}u\right)\check{\chi}_q(u) - \check{\chi}_q\left(q^{-2}u\right)\chi_q(u)$.

Orthogonal polynomials associated to $\boldsymbol{\chi}_q(u,\varepsilon)$

$$\boldsymbol{\chi}_q(u,\varepsilon)=\sum_{n\geq 0}\frac{\chi_{q,n}(\varepsilon)}{(q^{-2};q^{-2})_n}u^n=\sum_{n\geq 0}(-1)^n q^{n(n+1)}\frac{\chi_{q,n}(\varepsilon)}{(q^2;q^2)_n}u^n\,.$$

with polynomials $\chi_{q,n}(\varepsilon)\in\mathbb{C}[\varepsilon]$ satisfying the *recurrence relation*

$$\chi_{q,0}(\varepsilon)=1\,,\quad \chi_{q,n+1}(\varepsilon)=\varepsilon\chi_{q,n}(\varepsilon)+(q^n-q^{-n})^2\chi_{q,n-1}(\varepsilon),$$

with few first polynomials

$$\chi_{q,1}(\varepsilon)=\varepsilon,\quad \chi_{q,2}(\varepsilon)=\varepsilon^2+(q-q^{-1})^2,$$
$$\chi_{q,3}(\varepsilon)=\varepsilon(\varepsilon^2+(q^2-q^{-2})^2+(q-q^{-1})^2),\quad \dots$$

Multiplication rule

$$\chi_{q,m}(\varepsilon)\chi_{q,n}(\varepsilon)$$
$$=\sum_{k=0}^{\min(m,n)}\frac{(q^{2m};q^{-2})_k(q^{2n};q^{-2})_k(q^{2(k-m-n)};q^2)_k}{(q^2;q^2)_k}\chi_{q,m+n-2k}(\varepsilon)$$

The main functional equation with q replaced by q^{-1}

$$f(q^2u) + \frac{u^2}{q^2} f\left(\frac{u}{q^2}\right) = (1 - \varepsilon u + u^2) f(u).$$

There is no solution regular at $u = 0$. The series

$$\boldsymbol{\chi}_{q^{-1}}(u, \varepsilon) \simeq \sum_{n \geq 0} \frac{\chi_{q,n}(\varepsilon)}{(q^2; q^2)_n} u^n$$

does not converge, it is only an asymptotic expansion of the true solution

$$\boldsymbol{\chi}_{q^{-1}}(u, \varepsilon) := \frac{\check{\boldsymbol{\chi}}_q(u, \varepsilon)}{[\boldsymbol{\chi}_q, \check{\boldsymbol{\chi}}_q](u)}.$$

Behavior at infinity

In the limit $x \to -\infty$, equation

$$\psi(x + \mathrm{i}\mathsf{b}) + \psi(x - \mathrm{i}\mathsf{b}) = (\varepsilon - 2\cosh(2\pi \mathsf{b} x))\psi(x)$$

is approximated by the equation

$$\psi(x + \mathrm{i}\mathsf{b}) + \psi(x - \mathrm{i}\mathsf{b}) = -\,\mathrm{e}^{-2\pi \mathsf{b} x}\,\psi(x),$$

where, in the left hand side, any one of the two terms can be dominating giving rise to two possible asymtotics

$$\psi(x)|_{x\to-\infty} \sim \mathrm{e}^{\epsilon \mathrm{i}\pi x^2 + 2\pi\eta x}, \quad \epsilon \in \{\pm 1\}, \quad \eta := (\mathsf{b} + \mathsf{b}^{-1})/2.$$

Thus, there are two solutions of the form

$$\psi_\epsilon(x) = \mathrm{e}^{\epsilon \mathrm{i}\pi x^2 + 2\pi\eta x}\,\phi_\epsilon(x), \quad \phi_\epsilon(x)|_{x\to-\infty} = \mathcal{O}(1), \quad \epsilon \in \{\pm 1\}.$$

with the identification $\phi_\epsilon(x) = \boldsymbol{\chi}_{q^{-\epsilon}}(u)\overline{\boldsymbol{\chi}_{q^{\epsilon}}(u)}$.
The general, exponentially decaying at $x \to -\infty$, solution is of the form

$$\psi(x) = \mathrm{e}^{2\pi\eta x} \sum_{\epsilon\in\{\pm 1\}} A_\epsilon\,\mathrm{e}^{\epsilon \mathrm{i}\pi x^2}\,\phi_\epsilon(x)$$

Ansatz for the eigenfunction

$$\psi(x) := \mathsf{b}^{-1}\,\mathsf{e}^{\pi\mathsf{i}\sigma^2-\xi\pi\mathsf{i}/4}\,\mathsf{e}^{2\pi\eta x+\mathsf{i}\pi x^2}\,\frac{\check{\boldsymbol{\chi}}_q(u)\overline{\boldsymbol{\chi}_q(u)}+\xi\boldsymbol{\chi}_q(u)\overline{\check{\boldsymbol{\chi}}_q(u)}}{\theta_1(su,q)\theta_1(s^{-1}u,q)}.$$

where

$$[\boldsymbol{\chi}_q,\check{\boldsymbol{\chi}}_q](u)=\varrho\theta_1(su,q)\theta_1(s^{-1}u,q),$$

$$\theta_1(u,q) := \frac{1}{\mathsf{i}}\sum_{n\in\mathbb{Z}}(-1)^n q^{(n+1/2)^2}u^{n+1/2},$$

with certain functions $s = s(\varepsilon, q)$, $\varrho = \varrho(\varepsilon, q)$, $s := \mathsf{e}^{2\pi\mathsf{b}\sigma}$, and the variable $\xi \in \{\pm 1\}$ is the *parity* of the eigenstate: $\psi(-x) = \xi\psi(x)$. The function is real $\overline{\psi(x)} = \psi(x)$ (thus modular invariant $\mathsf{b} \leftrightarrow \mathsf{b}^{-1}$) and exponentially decays at both infinities

$$|\psi(x)| \sim \mathsf{e}^{-2\pi\eta|x|}, \quad x \to \pm\infty.$$

The quantization condition

The quantization condition is the analyticity condition for $\psi(x)$ with complex x in the strip

$$S_{\mathsf{b}} := \left\{z \in \mathbb{C} \mid |\Im z| < \max(|\Re \mathsf{b}|, |\Re \mathsf{b}^{-1}|)\right\}.$$

Define

$$G_q(u,\varepsilon) := \frac{\chi_q(u,\varepsilon)}{\check{\chi}_q(u,\varepsilon)}, \quad G_q(u,\varepsilon)G_q(1/u,\varepsilon) = 1, \quad \forall u \in \mathbb{C}_{\neq 0}.$$

Theorem

Let $\varepsilon = \varepsilon(\sigma)$ be such that $[\chi_q, \check{\chi}_q](u) = \varrho\theta_1(su)\theta_1(s^{-1}u)$ for any $u \in \mathbb{C}$, and assume that $s \notin \pm q^{\mathbb{Z}}$ (recall that $s = s(\sigma) = \mathrm{e}^{2\pi \mathsf{b}\sigma}$). Then the eigenfunction $\psi(x)$ does not have poles in the strip S_{b} if the variable σ is such that $G_q(s,\varepsilon) = -\xi\overline{G_q(s,\varepsilon)}$. Moreover, in that case, $\psi(x)$ is an entire function on $\mathbb{C}$.