Painlevé functions, Fredholm determinants and combinatorics

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Some motivation:

Ising diagonal correlation is a Painlevé VI tau function [Jimbo, Miwa, '81]

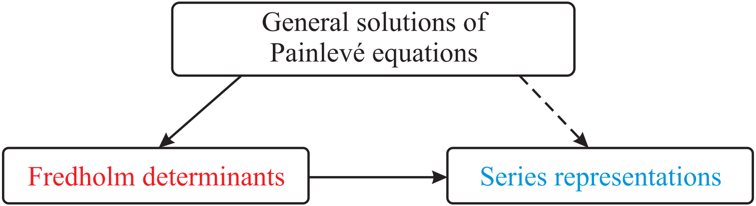
- $\langle \sigma(0,0)\,\sigma(N,N)\rangle \sim \tau_{\mathrm{VI}}(t)$
- t is a temperature parameter ($t = 0, 1, \infty \iff T = 0$, T_c, ∞)
- spin separation N is a parameter of PVI

Massive scaling limit:

- $T \to T_c$, $N \to \infty$, $r = N|T - T_c|$ finite
- Painlevé VI (classical) $\to$ Painlevé III (transcendental) [Wu, McCoy, Tracy, Barouch, '76]
- long-distance ($r \to \infty$): form factor expansions
- short-distance ($r \to 0$): conformal perturbation theory

What can be done for general Painlevé functions ?

- connection problems (FF $\leftrightarrow$ CPT)
- explicit representations (series, integrals, etc)
- ...



- block integrable kernels
- Widom's constants

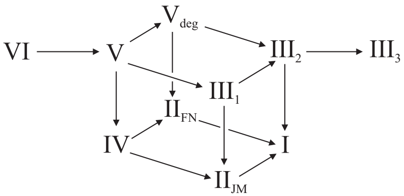
- summation over partitions/Young diagrams

Painlevé VI:

$$\left(t(t-1)\zeta''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

- $\zeta(t) = t(t-1)\frac{d}{dt}\ln\tau$, where $\tau(t)$ is Painlevé VI tau function

Musée des confluences:



	PVI	PV	PIII$_1$	PIII$_2$	PIII$_3$	PIV	PII	PI
$\sharp$(parameters)	4	3	2	1	0	2	1	0

(Special) solutions of Painlevé VI:

1. Hypergeometric Riccati family [Forrester, Witte, '02]

$$\tau_{\text{Riccati}}(t) = (1-t)^{-\frac{N\left(N+\nu+\nu'\right)}{2}} \det\Big[A_{j-k}(t)\Big]_{j,k=0}^{N-1},$$

$$A_m(t) = \frac{\Gamma(1+\nu')\, t^{\frac{\eta-m}{2}}\,(1-t)^{\nu}}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+\nu')}\, {}_2F_1\left[\begin{array}{c} -\nu, 1+\nu' \\ 1+\eta-m \end{array}\middle|\, \frac{t}{t-1}\right]$$
$$+\frac{\xi\,\Gamma(1+\nu)\, t^{\frac{m-\eta}{2}}\,(1-t)^{\nu'}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+\nu)}\, {}_2F_1\left[\begin{array}{c} 1+\nu, -\nu' \\ 1-\eta+m \end{array}\middle|\, \frac{t}{t-1}\right]$$

- PVI parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \frac{1}{2}(\eta, N, -N-\nu-\nu', \nu-\nu'+\eta)$ depend on $\nu, \nu', \eta \in \mathbb{C}$ and $N \in \mathbb{Z}_{\geq 0}$
- 1-parameter family of initial conditions depending on $\xi \in \mathbb{C}$

2. Elliptic Picard family [Kitaev, Korotkin, '98]

$$\tau_{\text{Picard}}(t) = \frac{e^{i\pi\sigma^2\bar{\tau}}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\, \frac{\vartheta_3(\sigma\pi\bar{\tau}+\sigma'\pi|\bar{\tau})}{\vartheta_3(0|\bar{\tau})}, \qquad \bar{\tau} = \frac{iK'(t)}{K(t)}$$

- PVI parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
- 2-parameter family of initial conditions depending on $\sigma, \sigma' \in \mathbb{C}$

3. Algebraic solutions

$$\tau_{H_3'}(t) = \frac{(1-s)^{\frac{1}{20}}\, s^{\frac{1}{20}}\, (1+3s)^{\frac{1}{12}}}{(1+s)^{\frac{3}{20}}\, (1-3s)^{\frac{11}{300}}\, (1+4s-s^2)^{\frac{1}{25}}},$$

$$t = \frac{(s-1)^5(3s+1)^3(s^2+4s-1)}{(s+1)^5(3s-1)^3(s^2-4s-1)}.$$

- $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \left(0, 0, 0, -\frac{1}{5}\right)$, no parameters in the initial conditions
- great icosahedron solution from [Dubrovin, Mazzocco, '98]

4. Fredholm determinant solutions [Borodin, Deift, '01]

$$\tau_{\mathrm{BD}}(t) = \det\left(\mathbf{1} - \lambda K\big|_{(0,t)}\right),$$

where continuous ${}_2F_1$ kernel $K(x,y) = \dfrac{\psi(x)\varphi(y) - \varphi(x)\psi(y)}{x-y}$ is defined by

$$\varphi(x) = x^{\theta_0}(1-x)^{\theta_1}\, {}_2F_1\left[\begin{matrix}\theta_0+\theta_1+\theta_\infty, \theta_0+\theta_1-\theta_\infty \\ 2\theta_0\end{matrix}; x\right],$$

$$\psi(x) = x^{1+\theta_0}(1-x)^{\theta_1}\, {}_2F_1\left[\begin{matrix}1+\theta_0+\theta_1+\theta_\infty, 1+\theta_0+\theta_1-\theta_\infty \\ 2+2\theta_0\end{matrix}; x\right].$$

- PVI parameters $(\theta_0, \theta_t = 0, \theta_1, \theta_\infty)$
- 1-parameter family of initial conditions depending on $\lambda \in \mathbb{C}$

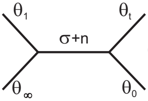
Question 1:

Can the general solution of Painlevé VI be expressed in terms of a Fredholm determinant?

General solution of PVI:

[Gamayun, Iorgov, OL, 1207.0787]

PVI tau function is a Fourier transform of $c = 1$ Virasoro conformal block:

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\, \mathcal{B}(\vec{\theta}, \sigma + n, t) = \sum_{n\in\mathbb{Z}} e^{in\eta}$$  (t)

- $\mathcal{B}(\vec{\theta}, \sigma, t) = t^{\alpha} \sum_{k=0}^{\infty} B_k(\vec{\theta}, \sigma)\, t^k$, with B_k rational in $\vec{\theta}, \sigma$ and determined by commutation relations of the Virasoro algebra
- as $c \to \infty$ (Virasoro $\to \mathfrak{sl}_2$), conformal block $\mathcal{B}(t) \sim {}_2F_1(t)$
- all 4 parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) \Longleftrightarrow$ external momenta
- 2 integration constants $(\sigma, \eta) \Longleftrightarrow$ internal momentum + Fourier conjugate variable

CFT derivations:

[Iorgov, OL, Teschner, 1401.6104]

- understood in the framework of Liouville CFT and generalized to an arbitrary number of punctures (Garnier system)
- uses quantum monodromy of conformal blocks with additional level 2 degenerate insertions

[Bershtein, Shchechkin, 1406.3008]

- bilinear differential-difference equations for conformal blocks coming from an embedding $\mathrm{Vir} \oplus \mathrm{Vir} \subset \mathrm{NSR} \oplus \mathrm{F}$
- extends to arbitrary values of central charge c

Extensions:

Irregular case (PV, PIV, PIII):

- [Gamayun, Iorgov, OL, 1302.1832], [Nagoya, 1505.02398]

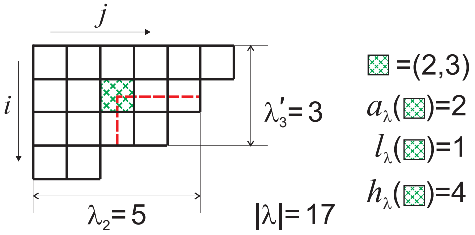
q-Painlevé:

- [Bershtein, Shchechkin, 1608.02566], [Jimbo, Nagoya, Sakai, 1706.01940]

AGT correspondence [Alday, Gaiotto, Tachikawa, '09]

$$\mathcal{B}(t) = \mathcal{Z}_{\text{inst}}(t) = \begin{array}{c}\text{combinatorial sum}\\ \text{over tuples of partitions}\end{array}$$ [Nekrasov, '04]

- coefficients of $\mathcal{B}(t)$ are explicit rational functions of parameters determined by geometry of appropriate Young diagram
- proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]
- provides series representation for general Painlevé VI function!



Young diagram associated to partition
$\lambda = \{6, 5, 4, 2\}$.

Conjecture [Gamayun, Iorgov, OL, 1207.0787]

Complete expansion of Painlevé VI tau function at $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\mathcal{B}(\vec{\theta}, \sigma + n; t),$$

where the function $\mathcal{B}(\vec{\theta}, \sigma; t)$ is explicitly given by

$$\mathcal{B}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\vec{\theta}, \sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$

$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$

Question 2:

How to understand this combinatorial structure without reference to CFT/gauge theory ?

$$\tau(t) \sim \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\sigma+n)\, t^{(\sigma+n)^2+|\lambda|+|\mu|}$$

Question 2:

How to understand this combinatorial structure without reference to CFT/gauge theory ?

$$\tau(t) \sim \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\sigma+n)\, t^{(\sigma+n)^2+|\lambda|+|\mu|}$$

Remark: Similar Fourier transform structure also appears for

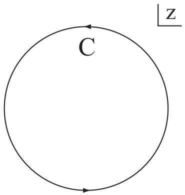
- regular type expansions for PV, $\mathrm{PIII}_{1,2,3}$ at $t=0$ (irregular CBs) [Gamayun, Iorgov, OL, 1302.1832]
- irregular expansions for PI–PV "nonlinear" Stokes rays at $t=\infty$, [Nagoya, 1505.02398], [Bonelli, OL, Maruyoshi, Sciarappa, Tanzini, 1612.06235]

Riemann-Hilbert setup

- let $\mathcal{C} \subset \mathbb{C}$ be a circle centered at the origin
- pick a loop $J(z) \in \mathrm{Hom}(\mathcal{C}, \mathrm{GL}_N(\mathbb{C}))$
- $J(z)$ continues into an annulus $\mathcal{A} \supset \mathcal{C}$

$$J(z) = \sum_{k \in \mathbb{Z}} J_k z^k,$$

- $\det J(z)$ has no winding along $\mathcal{C}$



Two Riemann-Hilbert problems:

$$\text{direct}: \quad J(z) = \Psi_-(z)^{-1} \Psi_+(z)$$

$$\text{dual}: \quad J(z) = \bar{\Psi}_+(z) \bar{\Psi}_-(z)^{-1}$$

Definition: The tau function of RHPs defined by $(\mathcal{C}, J)$ is defined as Fredholm determinant

$$\tau[J] = \det_{H_+}\left(\Pi_+ J^{-1} \Pi_+ J \Pi_+\right),$$

where $H = L^2\left(\mathcal{C}, \mathbb{C}^N\right)$ and Π_+ is the orthogonal projection on H_+ along H_-.

- operator $\Pi_+ J^{-1} \Pi_+ J \Pi_+$ is $\mathbf{1}$ + trace class
- dual RHP is solvable iff the operator $P := \Pi_+ J^{-1} \Pi_+$ is invertible on H_+, in which case $P^{-1} = \bar{\Psi}_+ \Pi_+ \bar{\Psi}_-^{-1} \Pi_+$
- likewise, for direct RHP and $Q := \Pi_+ J \Pi_+$, with $Q^{-1} = \Psi_+^{-1} \Pi_+ \Psi_- \Pi_+$
- if either direct or dual RHP is not solvable, then $\tau[J] = 0$

Remark: $\tau[J]$ appears [Widom, '76] in the asymptotics of determinant of block Toeplitz matrix with symbol J,

$$T_K[J] = \begin{pmatrix} J_0 & J_{-1} & J_{-2} & \dots & J_{-K+1} \\ J_1 & J_0 & J_{-1} & \dots & J_{-K+2} \\ J_2 & J_1 & J_0 & \dots & J_{-K+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{K-1} & J_{K-2} & J_{K-3} & \dots & J_0 \end{pmatrix}.$$

In this context, $\tau[J]$ is called Widom's constant.

Lemma: If the direct RHP is solvable, then $\tau[J]$ may also be written as

$$\tau[J] = \det_H \left(\mathbf{1} + K \right), \qquad K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \mathrm{End}\left(H_+ \oplus H_- \right),$$

where $\mathsf{a} : H_- \to H_+$, $\mathsf{d} : H_+ \to H_-$ are integral operators

$$(\mathsf{a} f)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}\left(z, z'\right) f\left(z'\right) dz', \quad (\mathsf{d} f)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{d}\left(z, z'\right) f\left(z'\right) dz'$$

with block integrable kernels

$$\mathsf{a}\left(z, z'\right) = \frac{\mathbf{1} - \Psi_+(z) \Psi_+\left(z'\right)^{-1}}{z - z'}, \qquad \mathsf{d}\left(z, z'\right) = \frac{\Psi_-(z) \Psi_-\left(z'\right)^{-1} - \mathbf{1}}{z - z'}.$$

Proof. We have $\mathsf{a} = \left(\Psi_+ \Pi_+ \Psi_+^{-1} - \Pi_+ \right)\Big|_{H_-}$, $\mathsf{d} = \left(\Psi_- \Pi_- \Psi_-^{-1} - \Pi_- \right)\Big|_{H_+}$. □

In our applications:

- $\Psi_\pm$ (direct factorization) are given and define the jump J
- $\Psi_\pm$ are expressed via classical special functions (Gauss, Kummer & Bessel for PVI, PV, PIII's)
- dual factorization ($\bar{\Psi}_\pm$) is a problem to be solved

Differentiation formula

Theorem: Let $(z,t) \mapsto J(z,t)$ be a smooth family of $\mathrm{GL}(N,\mathbb{C})$-loops which depend on an extra parameter t and admit both direct and dual factorization. Then

$$\partial_t \ln \tau[J] = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathrm{Tr}\left\{ J^{-1}\partial_t J \left[\partial_z \bar{\Psi}_- \, \bar{\Psi}_-^{-1} + \Psi_+^{-1}\, \partial_z \Psi_+ \right] \right\} dz.$$

- due to [Widom, '74]; rediscovered by [Its, Jin, Korepin, '06]
- related to (in general non-closed!) 1-form used by Malgrange, Bertola...

Corollary:

Widom's constant $\tau[J]$ $\simeq$ isomonodromic tau function

Isomonodromic example I

Fuchsian system with 4 regular singularities at $0, t, 1, \infty$:

$$\partial_z \Phi = \Phi A(z), \qquad A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}$$

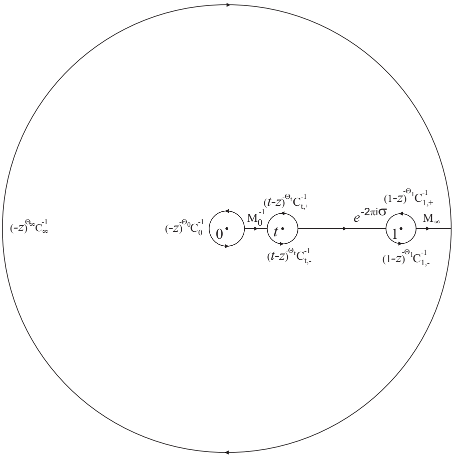
- arbitrary rank: $A_{0,t,1} \in \mathrm{Mat}_{N\times N}(\mathbb{C})$
- generic case: $A_{0,t,1}$ and $A_\infty := -A_0 - A_t - A_1$ are diagonalizable
- fix the diagonalizations $A_\nu = G_\nu^{-1} \Theta_\nu G_\nu$ with diagonal Θ_ν
- eigenvalues of A_ν are assumed distinct mod $\mathbb{Z}$

There exist unique fundamental solutions $\Phi^{(\nu)}(z)$, holomorphic on the universal covering of $\mathbb{C}\backslash\{0, t, 1\}$ and such that

$$\Phi^{(\nu)}(z) = \begin{cases} (\nu - z)^{\Theta_\nu} G^{(\nu)}(z), & \text{for } \nu = 0, t, 1, \\ (-z)^{-\Theta_\infty} G^{(\infty)}(z), & \text{for } \nu = \infty, \end{cases}$$

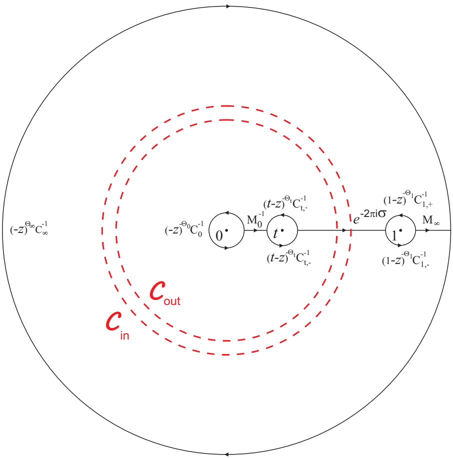
where $G^{(\nu)}(z)$ is holomorphic and invertible in a finite open disk around $z = \nu$ and satisfies $G^{(\nu)}(\nu) = G_\nu$.

Dual RHP I_1 for $\tilde{\Psi}$



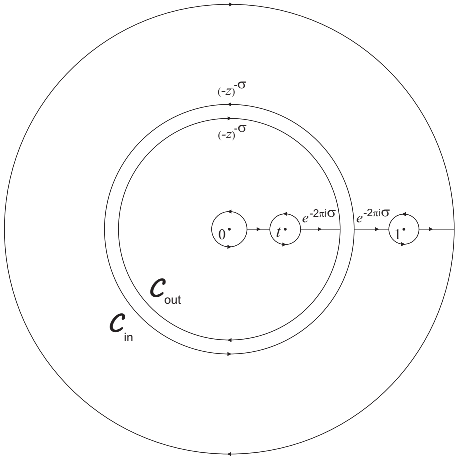
$$\tilde{\Psi}(z) = \begin{cases} G^{(\nu)}(z), & z \in D_\nu, \\ \Phi(z), & z \notin \mathbb{R}_{\geq 0} \cup \bar{D}_0 \cup \bar{D}_t \cup \bar{D}_1 \cup \bar{D}_\infty. \end{cases}$$

Dual RHP I_1 for $\tilde{\Psi}$

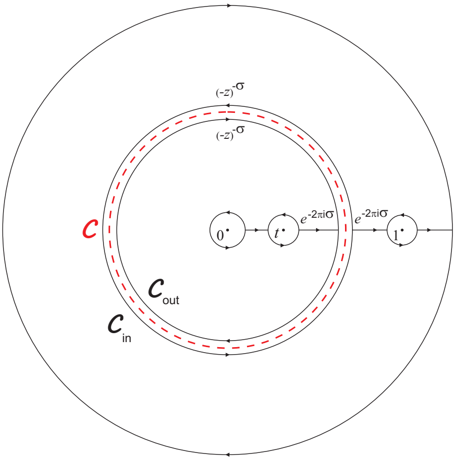


$$\hat{\Psi}(z) = \begin{cases} (-z)^{-\mathfrak{S}}\, \tilde{\Psi}(z), & z \in \mathcal{A}, \\ \tilde{\Psi}(z), & z \notin \bar{\mathcal{A}}. \end{cases}$$

Dual RHP I_2 for $\hat{\Psi}$

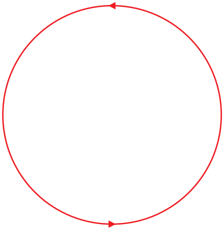


Dual RHP I_2 for $\hat{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

Dual RHP I_3 for $\bar{\Psi}$



$$\bar{\Psi}(z) = \begin{cases} \Psi_+(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi_-(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

- contour $\mathcal{C}$ (single circle !), smooth jump $J : \mathcal{C} \to \mathrm{GL}(N, \mathbb{C})$ given by

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \bar{\Psi}_+(z)\bar{\Psi}_-(z)^{-1}$$

- we are in the previously described setup!

Widom's differentiation formula

$$\partial_t \ln \tau\left[J\right] = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{Tr}\left\{J^{-1}\partial_t J\left[\partial_z \bar{\Psi}_- \, \bar{\Psi}_-^{-1} + \Psi_+^{-1}\,\partial_z \Psi_+\right]\right\} dz.$$

implies that

$$\partial_t \ln \tau\left[J\right] = \underbrace{\frac{\mathrm{Tr}\, A_0 A_t}{t} + \frac{\mathrm{Tr}\, A_t A_1}{t-1}}_{\partial_t \ln \tau_{\mathrm{JMU}}(t)} - \frac{\mathrm{Tr}\, A_0^+ A_t^+}{t},$$

so that in turn

$$\tau_{\mathrm{JMU}}\left(t\right) = t^{\frac{1}{2}\,\mathrm{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right)} \tau\left[J\right].$$

- Recall that

$$\tau\left[J\right] = \det\left(\mathbf{1} + K\right), \qquad K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix},$$

$$\mathsf{a}\left(z, z'\right) = \frac{\mathbf{1} - \Psi_+\left(z\right)\Psi_+\left(z'\right)^{-1}}{z - z'}, \qquad \mathsf{d}\left(z, z'\right) = \frac{\Psi_-\left(z\right)\Psi_-\left(z'\right)^{-1} - \mathbf{1}}{z - z'}.$$

- $\tau_{\mathrm{JMU}}\left(t\right)$ for 4-point system written via auxiliary 3-point solutions
- hypergeometric representations for $N = 2 \Longrightarrow$ Painlevé VI tau function !

For $N = 2$:

$$a(z,z') = \frac{(1-z')^{2\theta_1}\begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z)\end{pmatrix}\begin{pmatrix} K_{--}(z') & -K_{+-}(z') \\ -K_{-+}(z') & K_{++}(z')\end{pmatrix} - \mathbf{1}}{z-z'},$$

$$d(z,z') = \frac{\mathbf{1} - \left(1-\frac{t}{z'}\right)^{2\theta_t}\begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z)\end{pmatrix}\begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{+-}(z') \\ -\bar{K}_{-+}(z') & \bar{K}_{++}(z')\end{pmatrix}}{z-z'},$$

with

$$K_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix}\theta_1+\theta_\infty\pm\sigma, \theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma\end{matrix};z\right],$$

$$K_{\pm\mp}(z) = \pm\frac{\theta_\infty^2-(\theta_1\pm\sigma)^2}{2\sigma(1\pm 2\sigma)}\, z\, {}_2F_1\left[\begin{matrix}1+\theta_1+\theta_\infty\pm\sigma, 1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma\end{matrix};z\right],$$

$$\bar{K}_{\pm\pm}(z) = {}_2F_1\left[\begin{matrix}\theta_t+\theta_0\mp\sigma, \theta_t-\theta_0\mp\sigma \\ \mp 2\sigma\end{matrix};\frac{t}{z}\right],$$

$$\bar{K}_{\pm\mp}(z) = \mp t^{\mp 2\sigma}e^{\mp i\eta}\frac{\theta_0^2-(\theta_t\mp\sigma)^2}{2\sigma(1\mp 2\sigma)}\,\frac{t}{z}\, {}_2F_1\left[\begin{matrix}1+\theta_t+\theta_0\mp\sigma, 1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma\end{matrix};\frac{t}{z}\right].$$

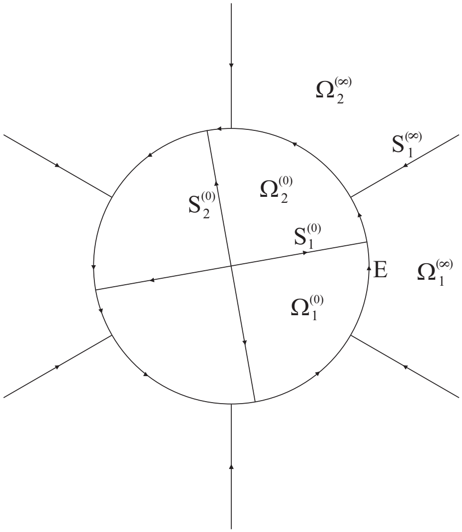
Isomonodromic example II

System with 2 irregular singularities at $0, \infty$:

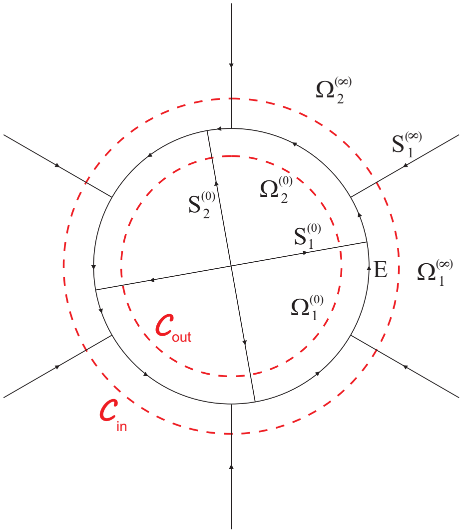
$$\partial_z \Phi = \Phi A(z), \qquad A(z) = \sum_{k=-R_0}^{R_\infty} z^{k-1} A_k$$

- $A_k \in \mathrm{Mat}_{N\times N}(\mathbb{C})$ and $\mathrm{Tr}\, A_k = 0$
- A_{-R_0}, A_{R_∞} are assumed diagonalizable
- canonical solutions in different sectors around $0, \infty$

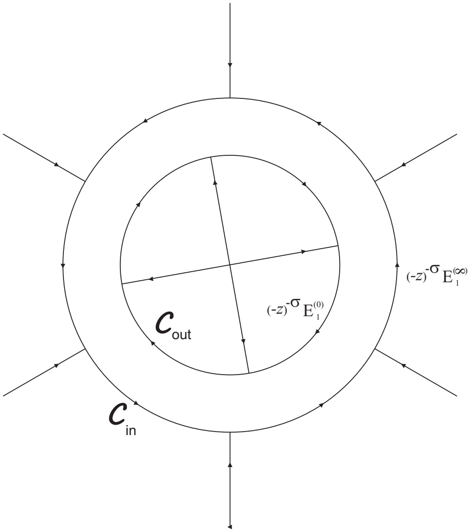
Dual RHP II_1 for $\tilde{\Psi}$



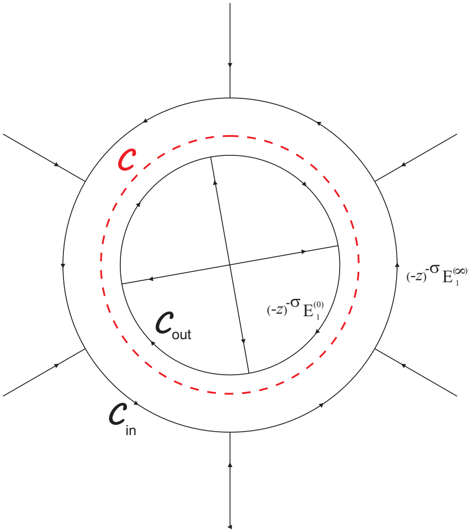
Dual RHP II_1 for $\tilde{\Psi}$



Dual RHP II_2 for $\hat{\Psi}$



Dual RHP II_2 for $\hat{\Psi}$



Again RHP on a single circle $\mathcal{C}$ with the jump

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z) = \bar{\Psi}_+(z)\,\bar{\Psi}_-(z)^{-1}.$$

Widom's differentiation formula implies that

$$\tau[J] = \frac{\tau_{\mathrm{JMU}}(\mathcal{T})}{\tau^{(0)}_{\mathrm{JMU}}\left(\mathcal{T}^{(0)}\right)\tau^{(\infty)}_{\mathrm{JMU}}\left(\mathcal{T}^{(\infty)}\right)}.$$

- $\tau^{(0,\infty)}_{\mathrm{JMU}}$ are Jimbo-Miwa-Ueno tau functions of auxiliary systems with one irregular and one regular singularity.
- dependence on 0-times and ∞-times is factorized in the denominator

von Koch formula

Given $K \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$, we can expand Fredholm determinant

$$\det(\mathbf{1}+K) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det K_{\mathfrak{Y}} = 1 + \sum_{m\in\mathfrak{X}} K_{mm} + \frac{1}{2!}\sum_{m,n\in\mathfrak{X}} \begin{vmatrix} K_{mm} & K_{mn} \\ K_{nm} & K_{nn} \end{vmatrix} + \dots$$

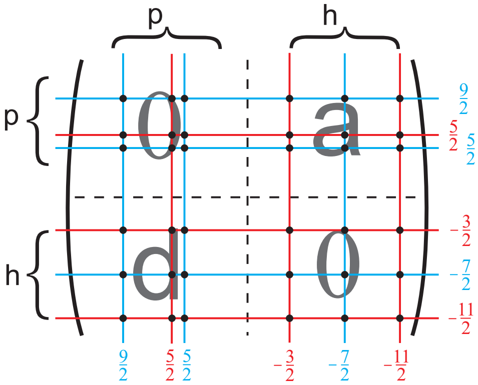
- in the Fourier basis,

  $$\mathsf{a}(z,z') = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{a}_{-q}^{\,p} z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}, \quad \mathsf{d}(z,z') = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{d}_{\,p}^{-q} z^{-\frac{1}{2}-q} z'^{-\frac{1}{2}-p},$$

  with $\mathsf{a}_{-q}^{\,p}, \mathsf{d}_{\,p}^{-q} \in \mathrm{Mat}_{N\times N}(\mathbb{C})$.
- multi-indices $m, n, \dots$ of principal minors

  $$\det K_{\mathfrak{Y}} = \det \begin{pmatrix} 0 & \mathsf{a}_{\mathsf{h}}^{\mathsf{p}} \\ \mathsf{d}_{\mathsf{p}}^{\mathsf{h}} & 0 \end{pmatrix}$$

  incorporate color indices $\alpha = 1, \dots N$ and (half-)integer Fourier indices

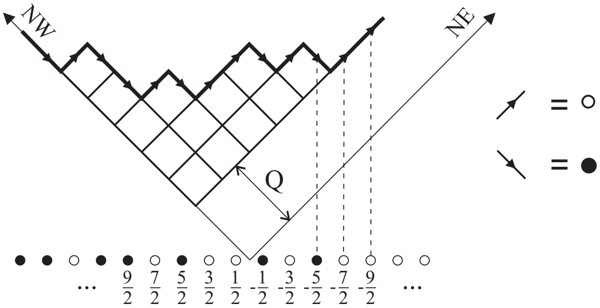


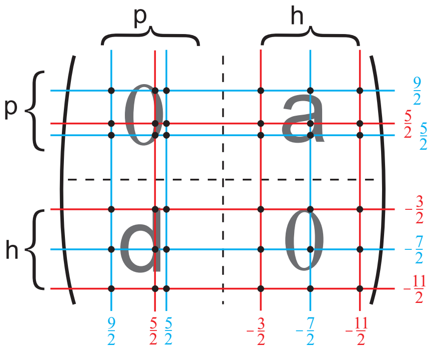
- combinatorial expansion

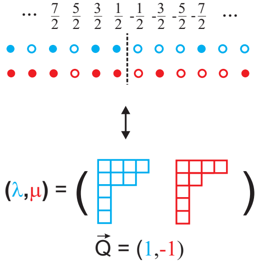
$$\det(\mathbf{1}+K)=\sum_{(\mathsf{p},\mathsf{h})}(-1)^{|\mathsf{p}|}\det \mathsf{a}_{\mathsf{h}}^{\mathsf{p}}\det \mathsf{d}_{\mathsf{p}}^{\mathsf{h}},$$

  with balance condition $|\mathsf{p}|=|\mathsf{h}|$
- Fourier indices in p and h are resp. positive and negative
- N subsets of $\mathbb{Z}'$

- A Maya diagram is a map $m : \mathbb{Z}' \to \{-1, 1\}$ subject to the condition $m(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- $\text{charge}(m) = \sharp(\text{particles}) - \sharp(\text{holes})$
- Maya diagram $=$ charged partition/Young diagram







- balanced configurations (p, h) are in bijection with N-tuples of Young diagrams of zero total charge
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q_\alpha = 0 \right\}.$$

- in the case $N = 2$, Widom's constant

$$\det(\mathbf{1} + K) = \sum_{(\lambda,\mu;n)\in\mathbb{Y}^2\times\mathbb{Z}} (-1)^{|\mathsf{p}|} \det \mathsf{a}_{\lambda,\mu,n} \det \mathsf{d}_{\lambda,\mu,n}$$

Isomonodromic examples

Explicit computation of elementary determinants $\det \mathsf{a}^{\mathsf{p}}_{\mathsf{h}}$, $\det \mathsf{d}^{\mathsf{h}}_{\mathsf{p}}$:

- a variant of Tracy-Widom conditions

$$\partial_z \Psi_\pm(z) = \Psi_\pm(z) A_\pm(z) + z^{-1} \Lambda_\pm(z) \Psi_\pm(z),$$

  with $A_\pm(z)$ rational in z and $\Lambda_\pm(z)$ polynomial in $z^{\pm 1}$.

- acting with $\mathcal{L}_0 = z\partial_z + z'\partial_{z'} + 1$ e.g. on

$$\frac{\mathbf{1} - \Psi_+(z)\Psi_+(z')^{-1}}{z - z'} = \sum_{p,q \in \mathbb{Z}'_+} \mathsf{a}_{-q}^{\ p} z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}$$

  yields a system of linear matrix equations on Fourier modes $\mathsf{a}_{-q}^{\ p}$ thanks to the fact that $\mathcal{L}_0 \frac{1}{z-z'} = 0$.

- PVI, V, III semisimple cases $(N = 2) \Longrightarrow$ Cauchy determinants

$$\det \frac{f_{p,\alpha} g_{q,\beta}}{p + q + \sigma_\alpha + \sigma_\beta}$$

- rewrite resulting factorized expressions using lengths of rows/columns instead of positions of particles/holes of different colors

Theorem [Gavrylenko, OL, 1608.00958]

Complete expansion of Painlevé VI tau function at $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\mathcal{B}(\vec{\theta}, \sigma + n; t),$$

where the function $\mathcal{B}(\vec{\theta}, \sigma; t)$ is explicitly given by

$$\mathcal{B}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\vec{\theta}, \sigma\right) t^{|\lambda|+|\mu|},$$

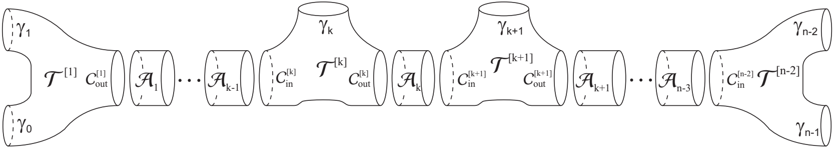
$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$

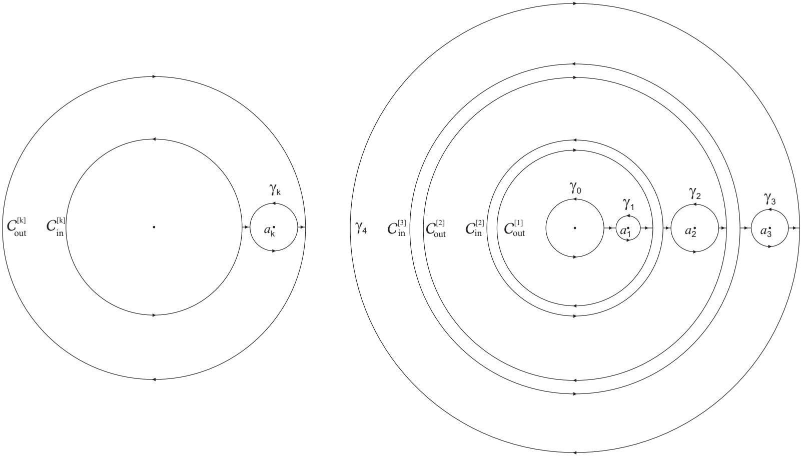
$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$

Garnier system

Auxiliary 3-point RHPs



- associate to the n-point RHP $n-2$ 3-point RHPs assigned to different trinions



Contour $\Gamma^{[k]}$ (left) and $\hat{\Gamma}$ for $n = 5$ (right)

- $\hat{\Psi}(z) = \begin{cases} \Psi(z) & \text{outside the annuli,} \\ (-z)^{-\mathfrak{S}_k} S_k^{-1} \Psi(z) & \text{inside.} \end{cases}$
- jumps on the boundary circles $\mathcal{C}_{k-1}^{\text{out}}$, $\mathcal{C}_k^{\text{in}}$ mimic regular singularities characterized by counterclockwise monodromies $M_{1\to k}$

Cauchy-Plemelj operators

- associate to every trinion $\mathcal{T}_k$ with $k = 2, \ldots, n-3$ the spaces of vector-valued functions

$$\mathcal{H}^{[k]} = \bigoplus_{\epsilon=\mathrm{in,out}} \left(\mathcal{H}^{[k]}_{\epsilon,+} \oplus \mathcal{H}^{[k]}_{\epsilon,-} \right), \qquad \mathcal{H}^{[k]}_{\epsilon,\pm} = \mathbb{C}^N \otimes \mathcal{V}_{\pm}\left(\mathcal{C}^{\epsilon}_k\right).$$

- elements $f^{[k]} \in \mathcal{H}^{[k]}$ will be written as

$$f^{[k]} = \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix}.$$

- define an operator $\mathcal{P}^{[k]} : \mathcal{H}^{[k]} \to \mathcal{H}^{[k]}$ by

$$\mathcal{P}^{[k]} f^{[k]}(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k \cup \mathcal{C}^{\mathrm{out}}_k} \frac{\Psi^{[k]}_{+}(z)\, \Psi^{[k]}_{+}(z')^{-1} f^{[k]}(z')\, dz'}{z - z'}$$

Lemma. We have $\left(\mathcal{P}^{[k]}\right)^2 = \mathcal{P}^{[k]}$ and $\ker \mathcal{P}^{[k]} = \mathcal{H}^{[k]}_{\mathrm{in},+} \oplus \mathcal{H}^{[k]}_{\mathrm{out},-}$. Moreover, $\mathcal{P}^{[k]}$ can be explicitly written as

$$\mathcal{P}^{[k]} : \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix} \mapsto \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a}^{[k]} & \mathsf{b}^{[k]} \\ \mathsf{c}^{[k]} & \mathsf{d}^{[k]} \end{pmatrix} \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix},$$

where the operators $\mathsf{a}^{[k]}$, $\mathsf{b}^{[k]}$, $\mathsf{c}^{[k]}$, $\mathsf{d}^{[k]}$ are defined by

$$\left(\mathsf{a}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{b}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{c}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k,$$

$$\left(\mathsf{d}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k.$$

- introduce the total space

$$\mathcal{H} := \bigoplus_{k=1}^{n-2} \mathcal{H}^{[k]}.$$

- there is a splitting

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

$$\mathcal{H}_\pm := \mathcal{H}^{[1]}_{\mathrm{out},\pm} \oplus \left(\mathcal{H}^{[2]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[2]}_{\mathrm{out},\pm}\right) \oplus \ldots \oplus \left(\mathcal{H}^{[n-3]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[n-3]}_{\mathrm{out},\pm}\right) \oplus \mathcal{H}^{[n-2]}_{\mathrm{in},\mp}.$$

- combine the 3-point projections $\mathcal{P}^{[k]}$ into an operator $\mathcal{P}_\oplus : \mathcal{H} \to \mathcal{H}$ given by the direct sum

$$\mathcal{P}_\oplus = \mathcal{P}^{[1]} \oplus \ldots \oplus \mathcal{P}^{[n-2]}.$$

- similarly, define another projection $\mathcal{P}_\Sigma : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{P}_\Sigma f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\Sigma} \frac{\hat{\Psi}_+(z)\, \hat{\Psi}_+(z')^{-1} f(z')\, dz'}{z - z'}, \quad \mathcal{C}_\Sigma := \bigcup_{k=1}^{n-3} \mathcal{C}_k^{\mathrm{out}} \cup \mathcal{C}_{k+1}^{\mathrm{in}}.$$

- it is easy to show that $\mathcal{P}_{\Sigma}\mathcal{P}_{\oplus} = \mathcal{P}_{\oplus}$ and $\mathcal{P}_{\oplus}\mathcal{P}_{\Sigma}=\mathcal{P}_{\Sigma}$
- the space
$$\mathcal{H}_{\mathcal{T}} := \operatorname{im}\mathcal{P}_{\oplus} = \operatorname{im}\mathcal{P}_{\Sigma}.$$
can be thought of as the subspace of functions on the union of boundary circles $\mathcal{C}_k^{\mathrm{in}}$, $\mathcal{C}_k^{\mathrm{out}}$ that can be continued inside $\bigcup_{k=1}^{n-2}\mathcal{T}_k$ with monodromy and singular behavior of the n-point fundamental matrix solution $\Phi(z)$
- varying the positions of singular points, one obtains a trajectory of $\mathcal{H}_{\mathcal{T}}$ in the infinite-dimensional Grassmannian $\mathrm{Gr}(\mathcal{H})$ defined with respect to the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$
- each of the subspaces $\mathcal{H}_{\pm}$ may be identified with $N(n-3)$ copies of the space $L^2(S^1)$ of functions on a circle; the factor $n-3$ corresponds to the number of annuli and N is the rank of the appropriate RHP

- introduce operators $\mathcal{P}_{\oplus,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ and $\mathcal{P}_{\Sigma,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ given by restrictions of $\mathcal{P}_\oplus$ and $\mathcal{P}_\Sigma$ to $\mathcal{H}_+$
- define $L \in \mathrm{End}\,(\mathcal{H}_+)$ defined by

$$L := {\mathcal{P}_{\oplus,+}}^{-1} \mathcal{P}_{\Sigma,+}$$

- there exists a basis in which $L^{-1} = \mathbf{1} - K$, with

$$K = \begin{pmatrix} U_1 & V_1 & 0 & . & 0 \\ W_1 & U_2 & V_2 & . & 0 \\ 0 & W_2 & U_3 & . & . \\ . & . & . & . & V_{n-4} \\ 0 & 0 & . & W_{n-4} & U_{n-3} \end{pmatrix}, \ \vec{g} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{n-3} \end{pmatrix}, \ \tilde{g}_k = \begin{pmatrix} g^{[k]}_{\mathrm{out},+} \\ g^{[k+1]}_{\mathrm{in},-} \end{pmatrix},$$

$$U_k = \begin{pmatrix} 0 & \mathrm{a}^{[k+1]} \\ \mathrm{d}^{[k]} & 0 \end{pmatrix}, \ V_k = \begin{pmatrix} \mathrm{b}^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \ W_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{c}^{[k+1]} \end{pmatrix}$$

Definition

The tau function associated to the Riemann-Hilbert problem for Ψ is defined as

$$\tau(a) := \det\left(L^{-1}\right)$$

Theorem

We have

$$\tau(a) = \Upsilon(a)^{-1}\tau_{\mathrm{JMU}}(a),$$

where $\tau_{\mathrm{JMU}}(a)$ is defined up to a prefactor independent of a by

$$d_a \ln \tau_{\mathrm{JMU}} = \sum_{1\leq k<l\leq n-1} \operatorname{Tr} A_k A_l \, d\ln(a_k - a_l),$$

and $\Upsilon(a) = \prod_{k=2}^{n-2} a_k^{\bar{\Delta}_k - \bar{\Delta}_{k-1} - \Delta_k}$, with $\Delta_k = \frac{1}{2}\operatorname{Tr}\Theta_k^2$, $\bar{\Delta}_k = \frac{1}{2}\operatorname{Tr}\mathfrak{S}_k^2$

Fourier basis

Let us represent the elements of $\mathcal{H}_{\mathcal{C}}$ by their Laurent series inside $\mathcal{A}$,

$$f(z) = \sum_{p\in\mathbb{Z}'} f^p z^{-\frac{1}{2}+p}, \qquad f^p \in \mathbb{C}^N,$$

and write integral kernels of 3-point projection operators $\mathrm{a}^{[k]}$, $\mathrm{b}^{[k]}$, $\mathrm{c}^{[k]}$, $\mathrm{d}^{[k]}$ as

$$\mathrm{a}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1} - \mathbf{1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{a}^{[k]}{}_{-q}^{\;p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}, \qquad z,z'\in\mathcal{C}_k^{\mathrm{in}},$$

$$\mathrm{b}^{[k]}(z,z') := -\frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{b}^{[k]}{}_{q}^{p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}-q}, \quad z\in\mathcal{C}_k^{\mathrm{in}}, z'\in\mathcal{C}_k^{\mathrm{out}}$$

$$\mathrm{c}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{c}^{[k]}{}_{-q}^{-p}\, z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}+q}, \; z\in\mathcal{C}_k^{\mathrm{out}}, z'\in\mathcal{C}_k^{\mathrm{in}}$$

$$\mathrm{d}^{[k]}(z,z') := \frac{\mathbf{1} - \Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{d}^{[k]}{}^{-p}_{\;\;q} z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}-q}, \quad z,z'\in\mathcal{C}_k^{\mathrm{out}}.$$

Principal minor

$$
\begin{pmatrix}
0 & \left(a^{[2]}\right)^{I_1}_{J_1} & \left(b^{[2]}\right)^{I_1}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
\left(d^{[1]}\right)^{J_1}_{I_1} & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & \left(a^{[3]}\right)^{I_2}_{J_2} & \left(b^{[3]}\right)^{I_2}_{I_3} & \cdot & \cdot & 0 & 0 \\
0 & \left(c^{[2]}\right)^{J_2}_{J_1} & \left(d^{[2]}\right)^{J_2}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \left(c^{[3]}\right)^{J_3}_{J_2} & \left(d^{[3]}\right)^{J_3}_{I_3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \left(b^{[n-3]}\right)^{I_{n-2}}_{I_{n-3}} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot & 0 & 0 & \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & \left(a^{[n-2]}\right)^{I_{n-3}}_{J_{n-3}} \\
0 & 0 & 0 & 0 & \cdot & \cdot & \left(c^{[n-3]}\right)^{J_{n-3}}_{J_{n-4}} & \left(d^{[n-3]}\right)^{J_{n-3}}_{I_{n-3}} & 0
\end{pmatrix}
$$

- vanishes unless balance condition $|I_k| = |J_k|$ is satisfied
- factorization into a product of elementary determinants

$$
Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right) := (-1)^{|I_k|}\det\begin{pmatrix} \left(a^{[k]}\right)^{I_{k-1}}_{J_{k-1}} & \left(b^{[k]}\right)^{I_{k-1}}_{I_k} \\ \left(c^{[k]}\right)^{J_k}_{J_{k-1}} & \left(d^{[k]}\right)^{J_k}_{I_k} \end{pmatrix}
$$

Corollary: Fredholm determinant $\tau(a)$ is given by

$$\tau(a) = \sum_{(\vec{I},\vec{J})\in \mathrm{Conf}_+} \prod_{k=1}^{n-2} Z_{I_k, J_k}^{I_{k-1}, J_{k-1}} \left(\mathcal{T}^{[k]}\right)$$

- The set Conf_+ of proper balanced configurations $\left(\vec{I}, \vec{J}\right)$ may be described in terms of Maya diagrams and charged partitions
- balanced configurations (I_k, J_k) are in one-to-one correspondence with N-tuples of Maya diagrams of zero total charge

Theorem

Fredholm determinant $\tau(a)$ can be written as a combinatorial series

$$\tau(a) = \sum_{\vec{Q}_{\mathbf{1}},\ldots\vec{Q}_{n-\mathbf{3}}\in\mathfrak{Q}_N} \sum_{\vec{Y}_{\mathbf{1}},\ldots\vec{Y}_{n-\mathbf{3}}\in\mathbb{Y}^N} \prod_{k=1}^{n-2} Z^{\vec{Y}_{k-\mathbf{1}},\vec{Q}_{k-\mathbf{1}}}_{\vec{Y}_k,\vec{Q}_k}\left(\mathcal{T}^{[k]}\right)$$

- elementary determinants $Z^{\vec{Y}_{k-\mathbf{1}},\vec{Q}_{k-\mathbf{1}}}_{\vec{Y}_k,\vec{Q}_k}$ are constructed from matrix elements of 3-point Plemelj operators in Fourier basis
- in rank $N = 2$, they are given by Cauchy matrices conjugated by diagonal factors $\Rightarrow$ explicitly computable !!!
- the result coincides with dual Nekrasov partition function for $U(2)$ linear quiver gauge theory with $\epsilon_1 + \epsilon_2 = 0$
- series representation for general solution of PVI/Garnier system
- rank $N \Rightarrow$ a sum of $N - 1$ Cauchy matrices
(unless additional spectral conditions are imposed)

Conclusions

1. A tau function (= Widom's constant) can be assigned to "any" RHP.
2. Given the direct factorization of the jump matrix, $\tau[J]$ may be written as a Fredholm determinant with a block integrable kernel.
3. Principal minor expansion of this determinant in the Fourier basis leads to combinatorial series over tuples of partitions.
4. In RHPs of isomonodromic origin, $\tau[J] \simeq \tau_{\mathrm{JMU}}(\mathcal{T})$
5. Kernels and coefficients of combinatorial series can be computed explicitly when auxiliary solutions from the direct factorization have hypergeometric representations; in particular, for Painlevé VI, V and III$_3$.
6. The procedure can be extended to the Garnier system.

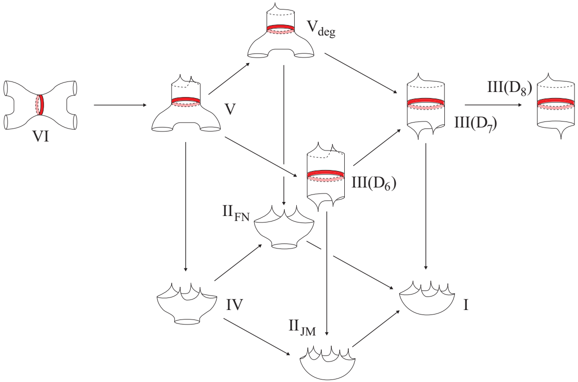
References

[1] P. Gavrylenko, OL, *Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions*, arXiv:1608.00958 [math-ph].

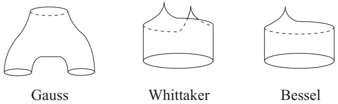
[2] P. Gavrylenko, OL, *Pure SU(2) gauge theory partition function and generalized Bessel kernel*, arXiv:1705.01869 [math-ph].

[3] M. Cafasso, P. Gavrylenko, OL, to appear.

Other Painlevé equations



Chekhov-Mazzocco-Rubtsov confluence diagram



Some solvable RHPs in rank $N = 2$