Yangian Symmetry for Fishnet Feynman Graphs

Florian Loebbert

Humboldt University Berlin

arXiv: 1704.01967 , *arXiv: 1708.00007* , work in progress
with Dmitry Chicherin, Vladimir Kazakov, Dennis Müller, De-liang Zhong

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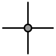
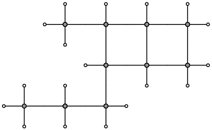
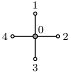
On the occasion of the 60th birthday of Jean-Michel Maillet.

Fishnet Feynman Graphs

Feynman graphs made from scalar four-point vertices in $4d$ $(x^2 = x^\mu x_\mu)$:

- Vertex  : $\int \mathrm{d}^4 x$
- Propagator j ——— k : $\frac{1}{x_{jk}^2} \equiv \frac{1}{(x_j - x_k)^2}$

e.g. 

Mostly unsolved, only cross known [Ussyukina '93, Davydychev]: 

$= \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$

Remarkable properties:

- Conformal symmetry (unbroken: no divergiencies)
- Represent integrable statistical vertex model [Zamolodchikov 1980]
- Here: New insights motivated by AdS/CFT integrability . . .

From $\mathcal{N} = 4$ SYM Theory to Fishnets

▶ Bi-Scalar Limit of Twisted $\mathcal{N}=4$ SYM

Planar $\mathcal{N}=4$ SYM
- integrable
- AdS/CFT dual

γ-Deformation
- 3 parameters γ_{jk}

Bi-scalar Model
- integrable?
- AdS/CFT dual?

$$\mathcal{L}_{\mathcal{N}=4} \xrightarrow{\phi_j\phi_k \to e^{-i\gamma_{jk}}\phi_j\phi_k} \mathcal{L}^{\gamma}_{\mathcal{N}=4} \xrightarrow[\xi\sim g e^{i\gamma}\ \text{fix}]{g\to 0,\ \gamma_{jk}\to i\infty} \mathcal{L}_{\text{bi}}$$

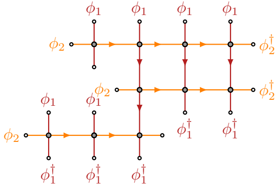
Resulting chiral, non-unitary, bi-scalar theory: [Gürdogan Kazakov 2015]

$$\mathcal{L}_{\text{bi}} = \mathrm{Tr}\left(\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + \xi^2\, \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\right).$$

Consider correlators

$$K = \langle \mathrm{tr}\left[\chi_1(x_1)\chi_2(x_2)\dots\chi_n(x_n)\right]\rangle, \qquad \chi_j \in \{\phi_1, \phi_2, \phi_1^\dagger, \phi_2^\dagger\}.$$

Each correlator given by single fishnet Feynman graph, e.g.



Where is the integrability of planar $\mathcal{N}=4$ SYM theory?

Integrability in $\mathcal{N}=4$ SYM Theory: Yangian

The Yangian algebra $Y[\mathfrak{g}]$ for the superalgebra $\mathfrak{g}=\mathfrak{psu}(2,2|4)$ underlies the integrability of the $\text{AdS}_5/\text{CFT}_4$ duality.

Yangian algebra $Y[\mathfrak{g}]$ (first realization): [Drinfel'd 1985]

Level 0 : $$\mathrm{J}^a=\sum_{k=1}^{n}\mathrm{J}^a_k\in\mathfrak{g}, \qquad [\mathrm{J}^a,\mathrm{J}^b]=f^{ab}{}_c\mathrm{J}^c$$

Level 1 : $$\widehat{\mathrm{J}}^a|_{1,n}=f^a{}_{bc}\sum_{j<k=1}^{n}\mathrm{J}^c_j\,\mathrm{J}^b_k, \qquad [\mathrm{J}^a,\widehat{\mathrm{J}}^b]=f^{ab}{}_c\widehat{\mathrm{J}}^c$$

Serre relations: $$[\widehat{\mathrm{J}}_a,[\widehat{\mathrm{J}}_b,\mathrm{J}_c]]-[\mathrm{J}_a,[\widehat{\mathrm{J}}_b,\widehat{\mathrm{J}}_c]]=\mathcal{O}(\mathrm{J}^3).$$

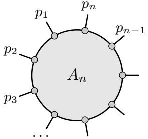
Yangian $Y[\mathfrak{psu}(2,2|4)]$ has been identified in $\mathcal{N}=4$ SYM theory for

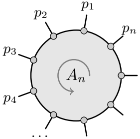
- Dilatation operator
- Scattering Amplitudes
- Wilson loops
- The action
- ...

[Dolan, Nappi Witten '03] [Drummond Henn, Plefka '09] [Mueller, Muenkler Plefka, Pollok Zarembo '13] [Beisert, Garus Rosso '17] [...]

Example: Color-Ordered Tree Amplitudes

Tree-level amplitudes in $\mathcal{N}=4$ SYM theory are cyclic functions:
[Drummond,Henn Korchemsky,Sokatchev, 2008][Drummond, Henn Plefka 2009]

$A_n =$ 

$=$ 

,

...and Yangian invariant:

$$\mathrm{J}^a A_n = 0, \qquad \widehat{\mathrm{J}}^a A_n = 0, \qquad \text{for } \mathrm{J}^a, \widehat{\mathrm{J}}^a \in Y[\mathfrak{psu}(2,2|4)]$$

Amplitude cyclic! Level-1 generator?:

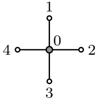
$$\widehat{\mathrm{J}}^a|_{2,n+1} - \widehat{\mathrm{J}}^a|_{1,n} = \underbrace{\tfrac{1}{2} f^a{}_{bc} f^{cb}{}_d}_{\text{dual Coxeter number zero for } \mathfrak{psu}(2,2|4)} - \underbrace{f^a{}_{bc} \mathrm{J}^c_1 \mathrm{J}^b}_{\text{annihilates amplitude}} \overset{[\text{on } A_n]}{=} 0 \quad \checkmark$$

Vanishing dual Coxeter number necessary for amplitude integrability?!

Cross Integral and $\mathfrak{so}(2,4)$

Consider the cross integral:

$$I_4 = \int \mathrm{d}^4 x_0 \, \frac{1}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} =$$



Yangian Level 0 (Conformal Lie Algebra Symmetry):

$$\mathrm{J}^a \begin{cases} \mathrm{D} &= -i x_\mu \partial^\mu - i\Delta, \\ \mathrm{L}_{\mu\nu} &= i x_\mu \partial_\nu - i x_\nu \partial_\mu, \\ \mathrm{P}_\mu &= -i\partial_\mu, \\ \mathrm{K}_\mu &= i x^2 \partial_\mu - 2i x_\mu x^\nu \partial_\nu - 2i\Delta x_\mu. \end{cases} \Bigg\} \in \mathfrak{so}(2,4)$$

For $\Delta = 1$ one finds: $\mathrm{J}^a I_4 = 0$ ✓

Cross Integral and $Y[\mathfrak{so}(2,4)]$

Consider the cross integral: $I_4 = \int \mathrm{d}^4 x_0 \frac{1}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2}$

Yangian Level 1?: Try! :

$$\widehat{\mathrm{P}}^{\mu}_{\mathsf{bi}} \simeq \sum_{j<k=1}^{n} \left[\left(\mathrm{L}_j^{\mu\nu} + \eta^{\mu\nu} \mathrm{D}_j \right) \mathrm{P}_{k,\nu} - (j \leftrightarrow k) \right] \quad \Rightarrow \quad \widehat{\mathrm{P}}^{\mu}_{\mathsf{bi}} I_4 = \sum_{j=1}^{4} j \mathrm{P}_j^{\mu} I_4$$

Employ Evaluation Representation:

guarantees cyclicity

$$\widehat{\mathrm{P}}^{\mu} = \widehat{\mathrm{P}}^{\mu}_{\mathsf{bi}} + \sum_{j=1}^{n} v_j \mathrm{P}_j, \qquad \text{with} \qquad v_j^{\mathsf{cross}} := -j$$

$$\Rightarrow \qquad \widehat{\mathrm{P}}^{\mu} I_4 = 0 \qquad \checkmark$$

- For $\mathfrak{so}(2,4)$ one level-one generator sufficient $\Rightarrow Y[\mathfrak{so}(2,4)]$
- Choice of v_j compensates for non-vanishing dual Coxeter number.

Is the Yangian symmetry of the cross an accident?

Generic Fishnets

RTT Realization of the Yangian

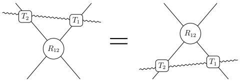
Collect generators into monodromy:

$$T(u) \simeq \mathbb{1} + \tfrac{1}{u}\mathrm{J} + \tfrac{1}{u^2}\widehat{\mathrm{J}} + \ldots,$$

Yangian algebra encoded in RTT-relations

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v)$$

with Yang's R-matrix $R_{12}(u-v) = \mathbb{1}_{12} + u\,\mathbb{P}_{12}$.



Monodromy is solution to RTT-relations:

$$T(\vec{u}) = L_n(u_n^+, u_n^-)L_{n-1}(u_{n-1}^+, u_{n-1}^-)\ldots L_1(u_1^+, u_1^-).$$

with conformal Lax operator:

inhomogeneities

differential conf. generators

$$L_{k,\alpha\beta}(u_k^+, u_k^-) = u_k\,\mathbb{1}_{k,\alpha\beta} + \underbrace{M_{\alpha\beta}^{ab}}_{[\Gamma_{6d}^a,\Gamma_{6d}^b]|_{\text{proj.}}}\,\mathrm{J}_{k,ab}^{\Delta_k}$$

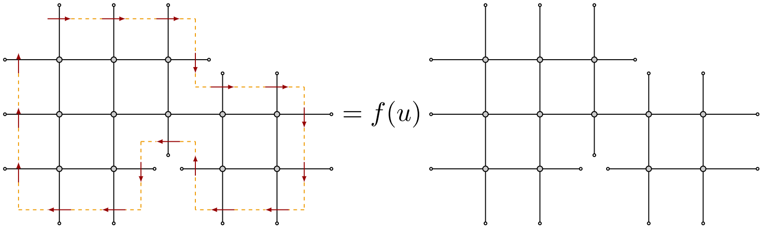
[Chicherin, Derkachov Isaev 2013]

and variables: $u_k^+ := u_k + \frac{\Delta_k - 4}{2}, \qquad u_k^- := u_k - \frac{\Delta_k}{2}.$

RTT and Yangian Invariants

Yangian invariance in the RTT formalism is encoded in the following eigenvalue equation for the n-point invariant I_n:

$$T(u)I_n = f(u)I_n \cdot \mathbb{1}, \qquad T(u) = f(u)\big(\mathbb{1} + \tfrac{1}{u}\mathrm{J} + \tfrac{1}{u^2}\widehat{\mathrm{J}} + \ldots.$$



Proof?: Pull monodromy through the graph! What are the rules?

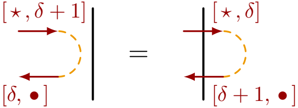
Related techniques employed in:

[Chicherin Kirschner '13] [Chicherin, Kirchner Derkachov '13] [Frassek, Kanning Ko, Staudacher '13] [Kanning, Lukowski Staudacher] [Broedel, de Leeuw Rosso '14] [Bargheer, Huang FL, Yamazaki '14] [Bork '15 Onishchenko] [Frassek, Meidinger Nandan, Wilhelm '15] [Bork '16 Onishchenko] [Frassek '16 Meidinger] [Ferro, Lukowski Orta, Parisi '16] [Fuksa '16 Kirschner] [Kirschner Savvidy '17] [...]

Intertwining Relations

1) Intertwiner:

$$\frac{1}{x_{12}^2} L_2[\delta, \bullet] L_1[\star, \delta+1] = L_2[\delta+1, \bullet] L_1[\star, \delta] \frac{1}{x_{12}^2}$$



2) Vacuum:

$$L_{\alpha\beta}[0,2] \cdot 1 = [2]\delta_{\alpha\beta} \qquad L^T_{\alpha\beta}[2,0] \cdot 1 = [2]\delta_{\alpha\beta}$$

(transposition in physical space (partial integration))

here we use shorthand notation:

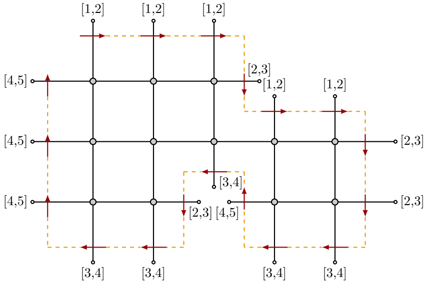
$$[\delta^+, \delta^-] := (u^+, u^-) \equiv (u+\delta^+, u+\delta^-), \qquad [\delta_k] := u + \delta_k.$$

Generic Fishnets

- The above relations allow to pull the monodromy through an arbitrary n-point scalar fishnet, thus proving its Yangian invariance:

$$T(u)I_n = L_n[\delta_n^+, \delta_n^-]L_{n-1}[\delta_{n-1}^+, \delta_{n-1}^-] \ldots L_1[\delta_1^+, \delta_1^-]I_n = f(u)I_n$$

- The inhomogeneities $\delta_k^\pm$ are attributed via a simple rule, e.g.



- Eigenvalue:

$$f(u) = \prod_{j \in \text{out}} (u + \delta_j^+)(u + \delta_j^-) \overset{\text{here}}{=} (u+3)^5(u+4)^5(u+4)^4(u+5)^4$$

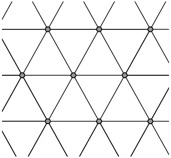
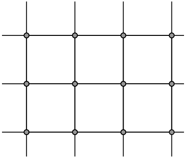
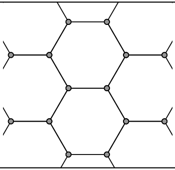
Generalizations

Different Dimensions

- In d dimensions the scalar propagator is $\int \mathrm{d}^d k \frac{e^{ikx}}{k^2} \sim |x|^{2-d}$.
- Intertwining relations generalize to $d = 3, 4, 6$:

$$\frac{1}{x_{12}^{d-2}} L_2^d[\delta, \bullet] L_1^d[\star, \delta + \tfrac{d-2}{2}] = L_2^d[\delta + \tfrac{d-2}{2}, \bullet] L_1^d[\star, \delta] \frac{1}{x_{12}^{d-2}}.$$

All Regular Tilings of the Plane give Yangian invariant integrals:

Dimension	$d = 3$	$d = 4$	$d = 6$
Propagator	$\lvert x_{ij} \rvert^{-1}$	$\lvert x_{ij} \rvert^{-2}$	$\lvert x_{ij} \rvert^{-4}$
Scalar Fishnet			

This is the complete set of fishnets introduced as statistical models by Zamolodchikov in 1980!

Are there more?

Including Fermions in 4d

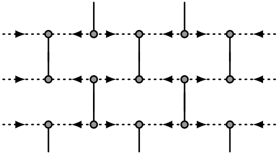
Obtain further integrable theories from limits of γ-deformed $\mathcal{N}=4$ SYM theory, e.g. two-fermion-three-scalar model: [Kazakov, Gurdogan Caetano 2016]

$$\mathcal{L}^{\text{int}}_{\phi\psi} = N_{\text{c}}\,\text{Tr}\left(\xi_1^2\phi_3^\dagger\phi_1^\dagger\phi^3\phi^1 + \xi_2^2\phi_2^\dagger\phi_1^\dagger\phi^2\phi^1 + \sqrt{\xi_1\xi_2}(\bar\psi_1\phi^1\bar\psi_4 - \psi^1\phi_1^\dagger\psi^4)\right).$$

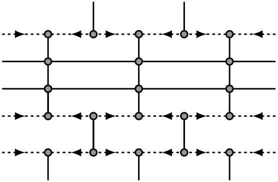
- Now we have 4pt scalar and 3pt Yukawa vertices.
- Use Lax operator for non-scalar representations [Chicherin, Derkachov Isaev 2013].

Yangian-invariant "Brick Wall" Feynman Graphs:

Solid



Unsolid



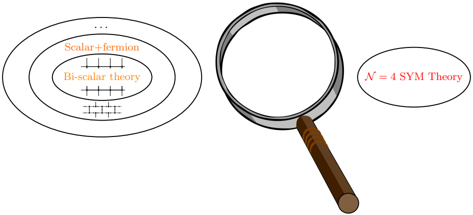
Summary and Outlook

Summary:

- Yangian symmetry (PDEs) for families of Feynman integrals.
- No supersymmetry needed!
- Novel "brick wall" classes of integrable fishnet graphs.
- Fishnets furnish a bridge between AdS/CFT integrability and indvidual Feynman integrals.

Outlook:

- Yangian symmetry of maximal limit-theory (3 bosons+3 fermions)?
- Develop new methods to compute Feynman graphs using integrability, c.f. [Basso Dixon 2017] [Gromov, Kazakov Korchemsky, Negro Sizov 2017]. . .



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