Yangian Symmetry for Fishnet Feynman Graphs

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On the occasion of the 60th birthday of Jean-Michel Maillet.
Fishnet Feynman Graphs

Feynman graphs made from scalar four-point vertices in $4d$ ($x^2 = x^\mu x_\mu$):

- **Vertex**
  \[
  \int d^4 x
  \]

- **Propagator**
  \[
  \frac{1}{x_{jk}^2} \equiv \frac{1}{(x_j - x_k)^2}
  \]

 Mostly unsolved, only cross known [Ussyukina '93 Davydychev]:

Remarkable properties:

- Conformal symmetry (unbroken: no divergencies)
- Represent integrable statistical vertex model [Zamolodchikov 1980]
- Here: New insights motivated by AdS/CFT integrability ...
From $\mathcal{N} = 4$ SYM Theory to Fishnets
Bi-Scalar Limit of Twisted $\mathcal{N} = 4$ SYM

Planar $\mathcal{N} = 4$ SYM
- integrable
- AdS/CFT dual

$\gamma$-Deformation
- 3 parameters $\gamma_{jk}$

Bi-scalar Model
- integrable?
- AdS/CFT dual?

\[ \mathcal{L}_{\mathcal{N}=4} \xrightarrow{\phi_j \phi_k \mapsto e^{-i\gamma_{jk} \phi_j \phi_k}} \mathcal{L}_{\mathcal{N}=4}^\gamma \xrightarrow{g \to 0, \gamma_{jk} \to i\infty} \xi \sim ge^{i\gamma} \text{ fix} \to \mathcal{L}_{\text{bi}} \]

Resulting chiral, non-unitary, bi-scalar theory: [Gürdogan Kazakov 2015]

\[ \mathcal{L}_{\text{bi}} = \text{Tr} \left( \partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + \xi^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \right) . \]

Consider correlators

\[ K = \langle \text{tr} \left[ \chi_1(x_1)\chi_2(x_2) \cdots \chi_n(x_n) \right] \rangle, \quad \chi_j \in \{ \phi_1, \phi_2, \phi_1^\dagger, \phi_2^\dagger \} . \]

Each correlator given by single fishnet Feynman graph, e.g.

Where is the integrability of planar $\mathcal{N} = 4$ SYM theory?
Integrability in $\mathcal{N} = 4$ SYM Theory: Yangian

The Yangian algebra $Y[\mathfrak{g}]$ for the superalgebra $\mathfrak{g} = \mathfrak{psu}(2, 2|4)$ underlies the integrability of the AdS$_5$/CFT$_4$ duality.

**Yangian algebra $Y[\mathfrak{g}]$ (first realization):** [Drinfel’d 1985]

\[
\text{Level 0}: \quad J^a = \sum_{k=1}^{n} J^a_k \in \mathfrak{g}, \quad \quad [J^a, J^b] = f^{abc} J^c
\]

\[
\text{Level 1}: \quad \hat{J}^a |_{1,n} = f^{abc}_{jk} \sum_{j<k} J^c_j J^b_k, \quad \quad [J^a, \hat{J}^b] = f^{abc} \hat{J}^c
\]

**Serre relations:**

\[
[\hat{J}_a, [\hat{J}_b, J_c]] - [J_a, [\hat{J}_b, \hat{J}_c]] = \mathcal{O}(J^3).
\]

Yangian $Y[\mathfrak{psu}(2, 2|4)]$ has been identified in $\mathcal{N} = 4$ SYM theory for

- Dilatation operator
- Scattering Amplitudes
- Wilson loops
- The action

[Dolan, Nappi Witten ’03] [Drummond Henn, Plefka ’09] [Mueller, Muenkler Plefka, Pollok Zarembo ’13] [Beisert, Garus Rosso ’17] […]
Example: Color-Ordered Tree Amplitudes

Tree-level amplitudes in $\mathcal{N} = 4$ SYM theory are cyclic functions:

\[
A_n = A_n \left( p_1, p_2, \ldots, p_n \right)
\]

\[
A_n \left( p_1, p_2, \ldots, p_n \right) = A_n \left( p_n, p_1, p_2, \ldots, p_{n-1} \right)
\]

\[
\ldots \text{and Yangian invariant:}
\]

\[
J^a A_n = 0, \quad \hat{J}^a A_n = 0, \quad \text{for } J^a, \hat{J}^a \in Y[\mathfrak{psu}(2, 2|4)]
\]

Amplitude cyclic! Level-1 generator?:

\[
\hat{J}^a \bigg|_{2,n+1} - \hat{J}^a \bigg|_{1,n} = \frac{1}{2} f^{a}_{bc} f^{cb}_{d} - f^{a}_{bc} J^c_1 J^b_1 \quad \text{[on } A_n] \quad \equiv \quad 0 \quad \checkmark
\]

Vanishing dual Coxeter number necessary for amplitude integrability?!
Consider the cross integral:

\[ I_4 = \int d^4 x_0 \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2} \]

**Yangian Level 0 (Conformal Lie Algebra Symmetry):**

\[
\begin{align*}
\mathbf{D} &= -ix_\mu \partial^\mu - i\Delta, \\
L_{\mu\nu} &= ix_\mu \partial_\nu - ix_\nu \partial_\mu, \\
P_\mu &= -i\partial_\mu, \\
K_\mu &= ix_2^2 \partial_\mu - 2ix_\mu x_\nu \partial_\nu - 2i\Delta x_\mu.
\end{align*}
\]

For \( \Delta = 1 \) one finds: \( J^a I_4 = 0 \) \( \checkmark \)
Cross Integral and $Y[\mathfrak{so}(2, 4)]$

Consider the cross integral: 
\[
I_4 = \int d^4 x_0 \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2}
\]

**Yangian Level 1?** Try! :

\[
\hat{P}_{bi}^\mu \approx \sum_{j<k=1}^{n} \left[ (L_{j}^{\mu\nu} + \eta^{\mu\nu} D_j) P_{k,\nu} - (j \leftrightarrow k) \right] \Rightarrow \hat{P}_{bi}^\mu I_4 = \sum_{j=1}^{4} j P_j^\mu I_4
\]

Employ Evaluation Representation:

\[
\hat{P}^\mu = \hat{P}_{bi}^\mu + \sum_{j=1}^{n} v_j P_j^\mu, \quad \text{with} \quad v_j^{\text{cross}} := -j
\]

\[
\Rightarrow \quad \hat{P}^\mu I_4 = 0 \quad \checkmark
\]

- For $\mathfrak{so}(2, 4)$ one level-one generator sufficient $\Rightarrow Y[\mathfrak{so}(2, 4)]$
- Choice of $v_j$ compensates for non-vanishing dual Coxeter number.

Is the Yangian symmetry of the cross an accident?
Generic Fishnets
RTT Realization of the Yangian

Collect generators into monodromy:

\[ T(u) \simeq 1 + \frac{1}{u}J + \frac{1}{u^2}\hat{J} + \ldots, \]

Yangian algebra encoded in RTT-relations

\[ R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v) \]

with Yang’s R-matrix \( R_{12}(u-v) = 1_{12} + u^{P_{12}} \).

Monodromy is solution to RTT-relations:

\[ T(\bar{u}) = L_n(u_n^+, u_n^-)L_{n-1}(u_{n-1}^+, u_{n-1}^-) \ldots L_1(u_1^+, u_1^-). \]

with conformal Lax operator:

\[ L_{k,\alpha\beta}(u_k^+, u_k^-) = u_k 1_{k,\alpha\beta} + M_{\alpha\beta}^{ab} J_{k,ab}^{\Delta_k} \]

[Chicherin, Derkachov, Isaev 2013]

and variables:

\[ u_k^+ := u_k + \frac{\Delta_k - 4}{2}, \quad u_k^- := u_k - \frac{\Delta_k}{2}. \]
Yangian invariance in the RTT formalism is encoded in the following eigenvalue equation for the $n$-point invariant $I_n$:

$$ T(u)I_n = f(u)I_n \cdot 1, \quad T(u) = f(u) \left( 1 + \frac{1}{u} J + \frac{1}{u^2} \hat{J} + \ldots \right) $$

**Proof?:** Pull monodromy through the graph! What are the rules?

Related techniques employed in:

- [Chicherin, Kirchner '13](#)
- [Chicherin, Kirchner, Derkachov '13](#)
- [Frass, Kanning, Ko, Staudacher '13](#)
- [Kanning, Lukowski, Staudacher](#)
- [Broedel, de Leeuw, Rosso '14](#)
- [Bargheer, Huang, FL, Yamazaki '14](#)
- [Bork '15, Onishchenko](#)
- [Frass, Meidinger, Nandan, Wilhelm '15](#)
- [Bork '16, Onishchenko](#)
- [Frass '16, Meidinger](#)
- [Ferro, Lukowski, Orta, Parisi '16](#)
- [Fuksa '16, Kirschner](#)
- [Kirschner, Savvidy '17](#)
- [Floth, Loebbert](#)
Intertwining Relations

1) **Intertwiner:**

\[
\frac{1}{x_{12}^2} L_2[\delta, \bullet] L_1[\star, \delta + 1] = L_2[\delta + 1, \bullet] L_1[\star, \delta] \frac{1}{x_{12}^2}
\]

\[
[\star, \delta + 1] \quad \Rightarrow \quad [\star, \delta]
\]

\[
[\delta, \bullet] \quad \Rightarrow \quad [\delta + 1, \bullet]
\]

2) **Vacuum:**

\[
L_{\alpha\beta}[0, 2] \cdot 1 = [2] \delta_{\alpha\beta}
\]

\[
L^{\hat{T}}_{\alpha\beta}[2, 0] \cdot 1 = [2] \delta_{\alpha\beta}
\]

Transposition in physical space (partial integration)

Here we use shorthand notation:

\[
[\delta^+, \delta^-] := (u^+, u^-) \equiv (u + \delta^+, u + \delta^-), \quad [\delta_k] := u + \delta_k.
\]
The above relations allow to pull the monodromy through an arbitrary \( n \)-point scalar fishnet, thus proving its Yangian invariance:

\[
T(u)I_n = L_n[\delta^+_n, \delta^-_n]L_{n-1}[\delta^+_n, \delta^-_{n-1}] \ldots L_1[\delta^+_1, \delta^-_1]I_n = f(u)I_n
\]

The inhomogeneities \( \delta^\pm_k \) are attributed via a simple rule, e.g.

\[
f(u) = \prod_{j \in \text{out}} (u + \delta^+_j)(u + \delta^-_j) \quad \text{here} \quad (u + 3)^5(u + 4)^5(u + 4)^4(u + 5)^4
\]
Generalizations
Different Dimensions

In $d$ dimensions the scalar propagator is \[ \int d^d k \frac{e^{i k \cdot x}}{k^2} \sim |x|^{2-d}. \]

Intertwining relations generalize to $d = 3, 4, 6$:

\[
\frac{1}{x_{12}^{d-2}} L_2^d[\delta, \bullet] L_1^d[\ast, \delta + \frac{d-2}{2}] = L_2^d[\delta + \frac{d-2}{2}, \bullet] L_1^d[\ast, \delta] \frac{1}{x_{12}^{d-2}}.
\]

All Regular Tilings of the Plane give Yangian invariant integrals:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>$d = 3$</th>
<th>$d = 4$</th>
<th>$d = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Propagator</td>
<td>$</td>
<td>x_{ij}</td>
<td>^{-1}$</td>
</tr>
<tr>
<td>Scalar Fishnet</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

This is the complete set of fishnets introduced as statistical models by Zamolodchikov in 1980!

Are there more?
Including Fermions in 4d

Obtain further integrable theories from limits of $\gamma$-deformed $\mathcal{N} = 4$ SYM theory, e.g. two-fermion-three-scalar model: [Kazakov, Gurdogan Caetano 2016]

$$\mathcal{L}^\text{int}_{\phi\psi} = N_c \text{Tr} \left( \xi_1^2 \phi_3^\dagger \phi_1^\dagger \phi_3^1 + \xi_2^2 \phi_2^\dagger \phi_1^\dagger \phi_2^1 + \sqrt{\xi_1 \xi_2} (\bar{\psi}_1 \phi_1^1 \bar{\psi}_4 - \psi_1^1 \phi_1^\dagger \psi_4) \right).$$

- Now we have 4pt scalar and 3pt Yukawa vertices.
- Use Lax operator for non-scalar representations [Chicherin, Derkachov Isaev 2013].

Yangian-invariant “Brick Wall” Feynman Graphs:
Summary and Outlook

Summary:
- Yangian symmetry (PDEs) for families of Feynman integrals.
- No supersymmetry needed!
- Novel “brick wall” classes of integrable fishnet graphs.
- Fishnets furnish a bridge between AdS/CFT integrability and individual Feynman integrals.

Outlook:
- Yangian symmetry of maximal limit-theory (3 bosons + 3 fermions)?
- Develop new methods to compute Feynman graphs using integrability, c.f. [Basso Dixon 2017] [Gromov, Kazakov Korchemsky, Negro Sizov 2017]...
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