# Understanding versus ignorance

## *Integrability, differential equations, zeros, universality*


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Knowledge can only arise from ignorance

If you want to learn what I understand read my papers. Today I will only discuss things which I do not understand.

Freeman Dyson

(at one of Joel Liebowitz's statistical mechanics conferences in the 1980's.)

# Acknowledgments

Much of this talk draws heavily on discussions with:

Alexander Abanov (Simons Center)

Michael Assis (Melbourne)

Jesper Jacobsen (Ecole Normale Superieur Paris)

Iwan Jensen (Melbourne)

Alexander Its (Purdue)

Jean-Marie Maillard (Paris 6 Jussieu)

Leonardo Rastelli (CN Yang Institute)

They are in no way responsible for my lack of understanding

# Outline

1. What is integrability?

   The 45 degree paradox

2. Differential equations

   Are we forever stuck at Painlevé VI?

3. The meaning of partition function zeros

   The tyranny of the Lee-Yang pinch

4. What is universality?

   Is most physics nonuniversal?

# 1. What is integrability?

The term “integrability” has been imported into the Euclidean statistical mechanics of lattice systems from the classical mechanics of dynamical many body problems and is used to refer to systems which have the property of commuting transfer matrices introduced by Baxter in 1969. However, Baxter himself never uses the term.

However, in classical dynamics time and space play different roles whereas in statistical mechanics all directions are the same. These differences are worth examining in detail.

1. In classical mechanics time is continuous and Hamilton's equations are first order in time derivatives. This restricts the analogue lattice statistical mechanical system in 2 ways:

a) The continuity in time is replaced by a translationally invariant interaction in one specified direction.

b) First order derivatives are replaced by nearest neighbor interactions in the specified direction.

2. No such restrictions need to be placed on the analogue of the spatial dimensions. However, if we want an isotropic model then interactions in the the spatial dimension must also be nearest neighbor and translationally invariant.

# Construction of models

With these restrictions we have only 2 properties to specify further for an integrable model

1. The degrees of freedom which go into the nearest neighbor translationally invariant transfer matrix. Various choices give Ising, 8 vertex, hard hexagon and chiral Potts.

2. The direction which is chosen to be the direction of transfer.

But, as pointed out by Baxter, models may be exactly solved which are not integrable. The most famous of these models is the Ising model.

# The 45 degree paradox

In 1968 TT Wu and I investigated layered Ising models where the vertical interactions are equal in a row but vary from row to row with a probability distribution. We used a row to row transfer matrix. The transfer matrices in different rows do not commute and the eigenvectors of thess matrices vary from row to row and we applied Furstenburg's theory of random matrix products (which is not the same thing as a product of random matrices).

The next year Rodney Baxter studied the row transfer matrix of similarly layered 6 vertex model which contains as a special case two decoupled Ising models rotated by 45 degrees. Baxter found that the eigenvectors did not depend on the layering and that the transfer matrices commuted. This discovery of Baxter lead to fame, fortune, Yang/Baxter equations and quantum groups.

Baxter’s methods of commuting transfer matrices and functional equations cannot be applied to the layered row Ising model.

Nevertheless for the Ising models the free energy, spontaneous magnetization and all correlation functions are computed without any use of the commutation properties of the 45 degree rotated transfer matrix. All correlations are given by determinants (in an infinite number of ways).

In particular both the diagonal $C(N,N)$ and the row $C(0,N)$ correlations are given as $N \times N$ Toeplitz determinants which look remarkably similar.

# Row/diagonal correlations

$$C(0,N) = \begin{vmatrix} a_0 & a_{-1} & \cdots & a_{-N+1} \\ a_1 & a_0 & \cdots & a_{-N+2} \\ \vdots & \vdots & & \vdots \\ a_{N-1} & a_{N-2} & \cdots & a_0 \end{vmatrix}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta}\left[\frac{(1-\alpha_1 e^{i\theta})(1-\alpha_2 e^{-i\theta})}{(1-\alpha_1 e^{-i\theta})(1-\alpha_2 e^{i\theta})}\right]^{1/2}$$

$$\alpha_1 = e^{-2E_v\beta}\tanh E_h\beta \qquad \alpha_2 = e^{-2E_h\beta}\coth E_v\beta$$

$C(N,N)$ is given by the determinant with

$$\alpha_1 = 0, \;\; \alpha_2 = (\sinh 2\beta E_v \sinh 2\beta E_h)^{-1}$$

# 2. Differential equations

The determinant for the diagonal correlation $C(N, N)$ has the remarkable property that it is the solution a (nonlinear) Painlevé VI equation. This was discovered by Jimbo and Miwa in 1981 which generalizes the result of Wu, McCoy, Tracy and Barouch who found in 1976 that in the scaling limit $C(N, N)$ satisfies a Painlevé III equation.

The relation (if any) of this result to integrability is unknown (at least to me).

# Painlevé for $C(N,N)$

For $C(N,N)$ define for $T < T_c$

$$t = \alpha_2^2 = (\sinh 2E_v\beta \sinh 2E_h\beta)^{-2}$$

$$\sigma_N(t) = t(t-1)\frac{d\ln C_-(N,N)}{dt} - \frac{t}{4}$$

and for $T > T_c$

$$t = 1/\alpha_2^2 = (\sinh 2E_v\beta \sinh 2E_h\beta)^{2}$$

$$\sigma_N(t) = t(t-1)\frac{d\ln C_+(N,N)}{dt} - \frac{1}{4}$$

$$\left(t(t-1)\frac{d^2\sigma}{dt^2}\right)^2$$

$$= N^2\left((t-1)\frac{d\sigma}{dt} - \sigma\right)^2 - 4\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt} - \sigma - \frac{1}{4}\right)\left(t\frac{d\sigma}{dt} - \sigma\right)$$

# Painlevé VI

This ODE for $C(N,N)$ is a special case of the general sigma form of the Painlevé VI equation

$$\frac{dh}{dt}\left(t(t-1)\frac{d^2}{dt^2}\right)^2 + \left[\frac{dh}{dt}\left(2h-(2t-1)\frac{dh}{dt}\right)+b_1b_2b_3b_4\right]^2$$
$$=\left(\frac{dh}{dt}+b_1^2\right)\left(\frac{dh}{dt}+b_2^2\right)\left(\frac{dh}{dt}+b_3^2\right)\left(\frac{dh}{dt}+b_4^2\right)$$

Since the discovery in 1981 of Painlevé VI for $C(N,N)$ and the earlier discovery of Painlevé III for the $T \to T_c$ scaled $C(N,N)$ there have been many physics applications of the six Painlevé nonlinear ODE's, particularly in random matrix theory.

# Painlevé Property

An ordinary differential equation or a completely integrable set of partial differential equations is said to have the Painlevé property if the locations of branch points and essential singularities do NOT depend on the boundary conditions. Are the six Painlevé equations very special or are there other nonlinear equations waiting to be discovered for other physics problems? The simplest generalization would seem to be $C(0, N)$.

The Painlevé equations are very special solutions of the Schlesinger equations of deformation theory and $C(0, N)$ does satisfy the deformation equations. However the complication that $\alpha_1 \neq 0$ has thus far imposed problems which have not been overcome.

# Fuchsian deformation theory

$$\frac{dY(z)}{dz} = \sum_{j=1}^{n} \frac{A_j}{z - a_j} Y(z)$$

The $A_j$ are $n \times n$ diagonalizable matrices. Monodromy preservation is equivalent to

$$\frac{\partial Y}{\partial a_j}(z) = -\frac{A_j}{z - a_j} Y(z) \quad j = 1, \cdots n$$

which is equivalent to a completely integrable system of non-linear differential (Schlesinger) equations

$$\frac{\partial A_j}{\partial a_k} = \frac{[A_j, A_k]}{a_j - a_k} \quad (j \neq k), \quad \frac{\partial A_j}{\partial a_j} = -\sum_{k \neq j} \frac{[A_j - A_k]}{a_j - a_k}$$

with the one form

$$\omega = d \ln \tau = \sum_{j<k} \mathrm{Tr} A_j A_k d \ln(a_k - a_k)$$

# Deformation and determinants

There is a theorem that an $N \times N$ Toeplitz determinant with a generating function $C(z)$ of product form

$$C(z) = \prod_{j=1}^{n} (z - a_j)^{\theta_j}$$

is the $\tau$ function of a $2 \times 2$ Fuchsian deformation equation with poles at $a_j$ and eigenvalues depending on $\theta_j$ and $N$.

For $C(N, N)$ the singularities may be set at $0, 1, t$ and $\infty$.

For $C(0, N)$ the singularities may be set at $0, \alpha_1, \alpha_2, \alpha_2^{-1}, \alpha_1^{-1}, \infty$

So $C(0, N)$ is the $\tau$ function.

# Okamoto

There is another theorem that the Schlesinger equations can be rewritten as a Hamiltonian system with the $a_j$ as the “times” with Hamiltonians $H_j$ depending on new variables $p_j, q_j$.

For $C(N, N)$ there is only one Hamiltonian and Okamoto was able to eliminate the auxiliary variables $p, q$ from the Hamiltonian equations to get the ODE for Painlevé VI.
This step is missing for $C(0, N)$.
Open questions:
1. How can nonlinear partial differential equations in $\alpha_1$ and $\alpha_2$ be found for $C(0, N)$?
2. Do ordinary non linear equations exist for $C(0, N)$?

# Inspiration from Myers

Painlevé equations (and possibly deformation theory itself) were introduced into physics in the 1964 paper of

John M. Myers Wave Scattering and geometry of a strip, J. Math. Phys. 6 (1965) 1839-1846. (astonishingly this paper has only 30 references in 52 years!).

This lead directly to the PIII result in 1976 for the scaled Ising correlation.

Myers has a second profound paper which is almost completely unknown

John M. Myers, Derivation of a matrix Painlevé equation germane to wave scattering from a broken corner, Physica D11 (1984) 51-89 (which has only 3 citations in 33 years!)

# Matrix Painlevé of Myers

The broken corner has conducting strips at $(0, t\sin\alpha)$ to $(0, +\infty)$ and $(t\cos\alpha, 0)$ to $(+\infty, 0)$.
Myers then finds an ordinary nonlinear equation in $t$

$$\left(\frac{\partial^2 w}{\partial t} + \frac{1}{t}\frac{\partial w}{\partial t}\right) - \frac{\partial w}{\partial t}w^{-1}\frac{\partial w}{\partial t} - wD(\alpha)wD(\alpha)w + D(\alpha)w^{-1}D(-\alpha) = 0$$

where $w$ is a $2 \times 2$ matrix and

$$D(\alpha) = \begin{bmatrix} \cos\alpha & 0 \\ 0 & \sin\alpha \end{bmatrix}$$

Note, if $D(\alpha)$ is replaced by $I$ and $w$ is diagonal this reduces to Painlevé III.

# Block Toeplitz determinants

Myers broken corner is surely a deformation problem with matrices larger than $2 \times 2$ even though Myers does not use the formalism of Jimbo, Miwa and Ueno. The question now arises: Are $n \times n$ block Toeplitz determinants related to some $m \times m$ matrix deformation problem which generalizes the relation of scalar Toeplitz determinants to $2 \times 2$ deformation problems. The most natural problem to investigate are correlations in the Ising model in the magnetic field $H/kT = i\pi/2$ where in B.M. McCoy and T.T. Wu, Phys. Rev 155 (1967) 438-452 the row correlation is shown to be a $2 \times 2$ block Toeplitz determinant with a matrix kernel which factorizes. The spontaneous magnetization and leading large separation behavior was computed. It is overwhelmingly probable that this is related to some very special matrix deformation problem.

# Myers for $C(M, N)$?

Myers broken corner problem is more general than the linear translational deformations used to obtain the PIII equation for scattering by a strip. This leads to the suggestion that deformation theory can be applied to formulations of Ising correlations which give determinants which are not Toeplitz. Indeed, while all Ising correlations can be written as determinants (in many equivalent ways) most of these determinants are not Toeplitz. Is the general correlation $C(M, N)$ the $\tau$ function of some $m \times m$ deformation problem with $m > 2$?

# Beyond isomonodromic deformation?

We finally need to acknowledge that while all equations coming from deformation theory have the Painlevé property there are many ODE's of order higher than two which have the Painlevé property which do not come from isomonodromic deformation. Perhaps the most famous of these is the Chazy III equation which has a natural boundary. This is the sort of equation which might characterize the Ising susceptibility.

# 3. Partition function zeros

Zeros of the partition function have been used to characterize phase transitions since the 1952 papers of Lee and Yang on the Ising model.

1. For finite sizes there are no zeros on either the positive temperature or positive fugacity axis.

2. In the thermodynamic limit the loci of zeros splits the temperature and/or fugacity plane into several disconnected regions. Each region corresponds to a different phase of the system.

3. In the zero free regions of the plane the free energy is analytic and correlations decay exponentially

# The tyranny of point pinches

In all cases studied numerically zeros pinch the positive temperature or fugacity axis only in points.

For temperature zeros this point defines a temperature $T_c$ where

1. Spontaneous magnetization sets in
2. The free energy has a singularity
3. The susceptibility has a singularity
4. Correlations are algebraic

The existence of a point pinch is needed for a field theory description.

Since Lee and Yang this picture dominates the scenario of second order phase transitions.

# Problems with pinches

1. There are actually NO studies of zeros using the variable $T$ or $\beta$ where the partition function is entire. Instead we use variables such as $x = e^{-2E/kT}$ which makes the partition function a polynomial. For the anisotropic Ising model this forces a restriction to $E_h/E_v$ being an integer.

2. The more significant problem is that there are important problems where point pinches may not occur but there may be line pinches instead.

The physics of rare events.

Long range interactions

# Physics of rare events

The Ising model with quenched random bonds is defined by

$$\mathcal{E} = -\sum_{j,k}\{E_h(j,k)\sigma_{j,k}\sigma_{j,k+1} + E_v(j,k)\sigma_{j,k}\sigma_{j+1,k}\}$$

where the bonds $E_h(j,k)$ and $E_v(j,k)$ are chosen randomly with a probability distribution.

There are two classic studies of this model done in the late 1960's.

1. The layered two dimensional model at $H = 0$ as a function of $T$.

2. The fully random model in 2 and 3 dimensions for fixed $T$ as a function of $H$.

These papers initiate the physics of rare events into statistical mechanics.

# Griffiths temperatures

The free energy of the random bond model in the thermodynamic limit approaches a unique limit with probability one. Of course, a periodic array of impurities will act like a translationally invariant lattice with a large unit cell. These configurations will have measure zero. Similarly fractal lattices built on a recursion scheme will have zero measure.

Let $E^U$ (and $E^L$) be the strongest (and weakest) bond strengths allowed by the probability distribution and let $T_c(E^U)$ and $T_c(E^L)$ be the critical temperatures the pure nonrandom lattice with interactions $E^U$ and $E^L$.

Question: Will there be zeros which pinch the entire line segment between $T_c(E^L)$ and $T_c(E^U)$?

# The layered model

B.M. McCoy and T.T. Wu, Theory of a two dimensional Ising model with random impurities I: Thermodynamics, Phys. Rev 176 (1968) 631-643; II spin correlation functions. Phys. Rev. 188 (1969) 982.

B.M. McCoy, Theory of a two two dimensional Ising model with random impurities III boundary effects, Phys. Rev. 188 (1969) 1014;IV, Phys. Rev. B2 (1970) 2795

The bulk specific heat has an infinitely differentiable singularity at the $T_c$ where spontaneous magnetization sets in.
The boundary row magnetic susceptibility diverges in a temperature region around $T_c$.
For $T_c(E^L) < T < T_c(E^U)$ the average row correlation decays with a temperature dependent power law.

# Site diluted fully random

R.B. Griffiths, Nonanalytic behavior above the critical temperature in a random Ising ferromagnet Phys. Rev. Letts. 23 (1969) 17-19.

$$P(E) = p\delta(E - E_0) + (1 - p)\delta(E)$$

Griffiths shows that for all $T$ below the $T_c(p = 1)$ of the pure case $p = 1$ that the magnetization $M(H)$ is not an analytic function of $H$ at $H = 0$ (even below $T_c$).

Giffiths also states without proof that there will be zeros for $T < T_c(p = 1)$.

This seems to be the first time that it is suggested that temperature zeros can pinch in line segments

# $T_c(E^L) < T < T_c(E^U)$

Layered model:

The known effects of randomness at $H = 0$ are sever but nothing in the bulk is known about analyticity of $M(H)$ at $H = 0$; The region for $T_c(E^L) < T < T_c(E^U)$ is a separate phase different from the ordered phase $T < T_c(E^L)$ and the disordered phase $T_c(E^U) < T$. Rare events are clearly very important.

Fully random model:

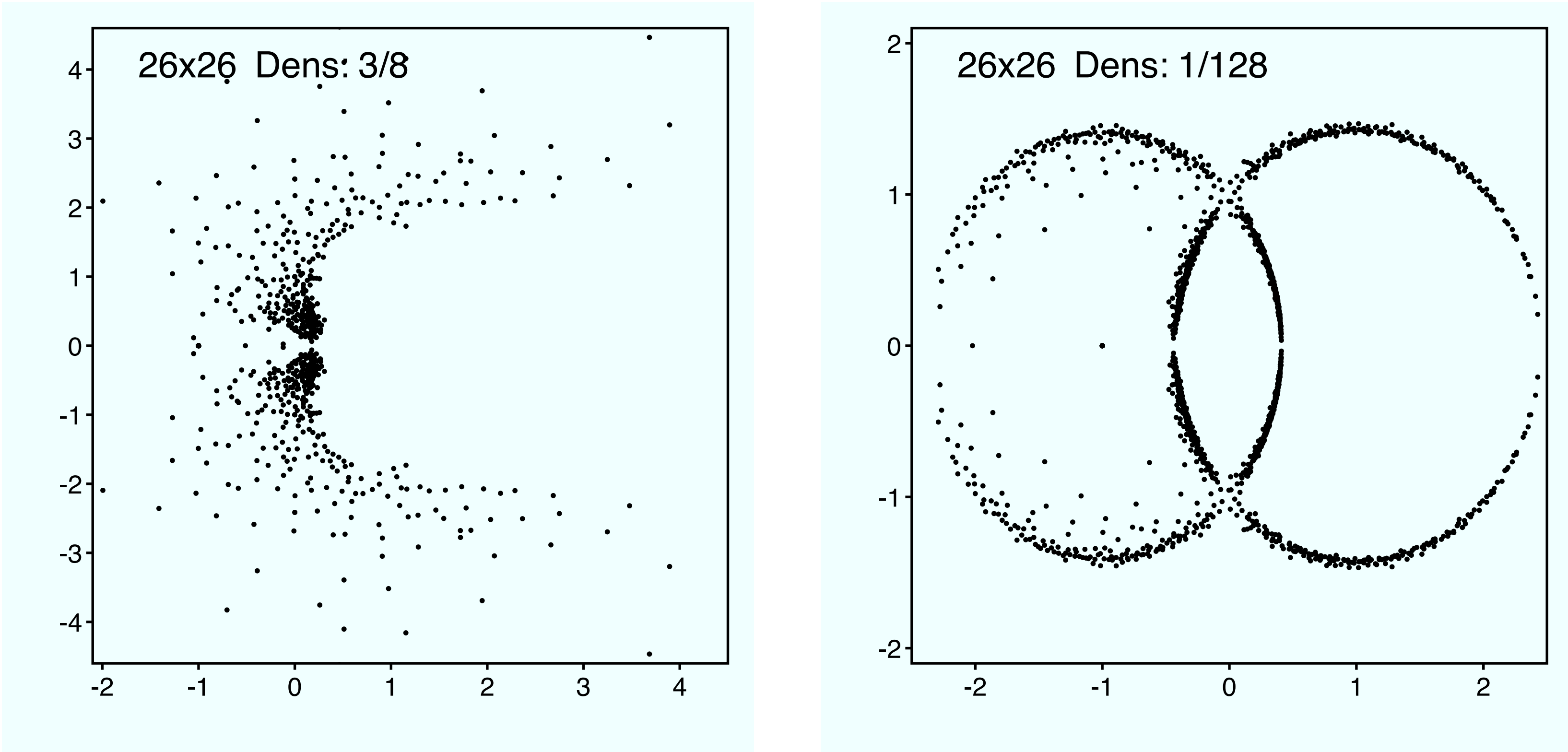
Subsequent to Griffiths' work it has been shown that for $T_c < T < T_c(E^U)$ the singularities in $M(H)$ are infinitely differentiable essential singularities. and at least if $T_c(E^U) = \infty$ the correlations decay exponentially. For $T_c(E^L) < T < T_c$ nothing beyond Griffiths seems to have been computed. The influence of rare events is far less clear than for the layered model.

# Random zeros

With probability one the limiting distribution of zeros in the thermodynamic limit of any one random collection of bonds will characterize the system. However. the partition function in the temperature variable while it is entire is not a polynomial and no computations of temperature zeros has ever been done. Instead the following two special cases which are polynomials in $z = e^{-2E_0/kT}$ can give insight.

1. $P(E) = p\delta(E) + (1-p)\delta(E - E_0)$ (This is the bond diluted case and has one of the Griffiths temperatures at zero)
2. $P(E) = p\delta(E - E_0) + (1-p)\delta(E - 2E_0)$

# From Iwan Jensen




Partition function zeros for the bond dilute model in the complex plane $z = e^{-E/kT}$ for $p = 3/8$ and $p = 1/128$ for a $26 \times 26$ lattice.

# An interpretation

These plots have the interpretation that in the thermodynamic limit zeros will fill an area in the complex $z$ plane which includes the segment $0 \leq z \leq \sqrt{2} - 1 = 0.414 \cdots$.

This is in distinct contrast with the field theory approach to randomness which is based on the scenario of a point pinch.

# Long range Ising model

The other model of interest is the long range Ising model in 2 dimensions defined by

$$\mathcal{E} = -\sum_{\mathbf{r}_j \neq \mathbf{r}_k} \frac{\sigma_{\mathbf{r}j}\sigma_{\mathbf{r}k}}{|\mathbf{r}_j - \mathbf{r}_k|^{2+s}}$$

This model is important to extend the Lee-Yang lattice gas model of the critical point from short (finite) range interactions to the long range attraction of a Lenard-Jones 6-12 potential.

The following properties have been known for decades:

1. For $T > T_c$ the correlations $\langle \sigma_0 \sigma_{\mathbf{r}} \rangle$ decay as $1/|\mathbf{r}|^{2+s}$

2. There are three regions of algebraic decay at $T = T_c$

$s > 7/4, \quad 1 < s < 7/4, \quad 0 < s < 1$

# Long range zeros

What is quite unknown is the behavior of the correlations for $T \neq T_c$. The algebraic decay of the correlations strongly suggests that there is a segment of zeros pinching the positive temperature axis. This in turn suggests that field theory methods are not sufficient to study $T \neq T_c$.

However, absolutely no computations of zeros have ever been done.

# 4. What is universality?

The standard and most restrictive definition of a universality class is that the singularities at $T_c$ are the same for all models in the class. As an example, for the long range Ising model, with $7/4 < s$ the long distance decay of the two point correlation is (believed) to be independent of $s$ and is the same as the decay of the correlation for the nearest neighbor model $C/r^{1/4}$.

Does any of this universality extend to $T \neq T_c$?

In particular does the multiparticle representation of the (scaled) Ising two point function of the nearest neighbor model found in 1976 and incorporated in PIII and PVI extend to the long range model with $7/4 < s$?

# An immediate problem

The first obstacle to a universal particle structure for the long range Ising model is that the two point function decays as $1/r^{2+s}$ instead of $e^{-r/\xi}/r^{1/2}$ as required for a particle interpretation.

There is a cheap way to argue this away by comparing the leading exponential decay of the nearest neighbor model near $T = T_c$ where $x = (T - T_c)r$ is order one

$$(T - T_c)^{1/4} \frac{e^{-x}}{x^{1/2}}$$

with the powerlaw decay

$$\frac{C(T)}{r^{2+s}} = (T - T_c)^{2+s} \frac{C(T)}{x^{2+s}}$$

# Computation of $C(T)$

$$\langle \sigma_0 \sigma_{\mathbf{r}} \rangle = Z^{-1} \sum_{\sigma=\pm 1} \sigma_0 \sigma_{\mathbf{r}} e^{\mathcal{E}_0/kT + \mathcal{E}_L/kT},$$

$$\mathcal{E}_0 = -E \sum_{j,k} \sigma_{j+1,k} \sigma_{j,k+1} \quad \mathcal{E}_L = -E \sum_{\mathbf{r}_j, \mathbf{r}_k}{}' \frac{1}{|\mathbf{r}_j - \mathbf{r}_k|^{2+s}} \sigma_{\mathbf{r}_j} \sigma_{\mathbf{r}_k}$$

Treating the term $e^{\mathcal{E}_L/kT}$ as a perturbation

$$\langle \sigma_0 \sigma_r \rangle = \langle \sigma_0 \sigma_r \rangle_0 + \sum_{\mathbf{r}_j, \mathbf{r}_k}{}' \frac{\langle \sigma_0 \sigma_r \sigma_{\mathbf{r}_j} \sigma_{\mathbf{r}_k} \rangle_0}{kT|\mathbf{r}_j - \mathbf{r}_k|^{2+s}}$$

The sum over $\mathbf{r}_j$ and $\mathbf{r}_k$ is dominated by $\mathbf{r}_j \sim 0$ and $\mathbf{r}_k \sim r$

$$\langle \sigma_0 \sigma_r \rangle \to \langle \sigma_0 \sigma_r \rangle_0 + \frac{1}{r^{2+s} kT} \left( \sum_{\mathbf{r}_j} \langle \sigma_0 \sigma_{\mathbf{r}_j} \rangle_0 \right)^2$$

$$C(T)/kT = \left( \sum_{\mathbf{r}_j} \langle \sigma_0 \sigma_{\mathbf{r}_j} \rangle_0 \right)^2 = \chi(T)^2 \to (T - T_c)^{-7/2}$$

The exponential and powerlaw terms are comparable when

$$\frac{e^{-x}}{x^{1/2}} = (T - T_c)^{(7/4+s)} \frac{C(T)}{x^{2+s}}$$

so that

$$x \sim \ln(T - T_c)^{-(7/4+s)} C(T)^{-1}$$

So if

$$\lim_{T \to T_c} C(T)(T - T_c)^{(7/4+s)} = 0$$

Then for any fixed $x$ the exponential dominates the power law decay. Using

$$C(T) \to (T - T_c)^{-7/2}$$

we conclude that the exponential dominates the scaling function for $7/4 < s$,

# Particles and universality

Outside of the scaling region for $T > T_c$ there is no exponential decay and obviously no universality with nearest neighbor Ising.

Universality for $T > T_c$ and $7/4 < s$ would mean that there is some correlation length which diverges at $T_c$ such that the scaled 2 point function is identical with the PIII scaling function of the nearest neighbor model. However, the restriction to $7/4 < s$ indicates that if there is a particle interpretation for $T > T_c$ and $7/4 < s$ it will breakdown at $s = 7/4$.

Even in the scaling region there seem to be no arguments to prevent the attractive long range force from producing bound states whose spectrum depends on $s$. This would be somewhat like bound states in the nearest neighbor model in the presence of a scaled magnetic field.

# Summary of ignorance

1. There are no massive lattice models where correlation functions have been computed using the methods of integrability.

2. The applications of deformation theory to statistical mechanics is grossly unexplored.

3. Studies of partition function zeros for random bond and long range Ising models are almost nonexistent.

4. We know very little about nonuniversal physics such as the spectrum of particles for $T \neq T_c$.

# Thank You

I will conclude by thanking the organizers for the opportunity of presenting these problems to the people best able to solve them.

One thing is certain. We are not going to run out of problems to solve.