Quantum separation of variables from the spectrum to the matrix elements of local operators of integrable quantum models

Correlation functions of quantum integrable systems and beyond, ENS-Lyon

On the occasion of the 60th birthday of Jean Michel Maillet

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Subject related to past and current collaborations over the last 9 years with:

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Main aims:

- To solve exactly lattice integrable quantum models by quantum separation of variables (SOV) characterizing both their spectrum and dynamics (time dependent correlation functions).
- To define a microscopic approach to solve exactly 1+1 dimensional quantum field theories (QFT) by using the SOV solution of their integrable lattice regularizations.

Original state of art of the Quantum Separation of Variables (SOV):

- The quantum version of SOV has been invented by E. Sklyanin (1985) and applied to some specific integrable quantum models (like Toda model and XXZ spin chains).
- SOV applied for few others integrable quantum models by some few key researchers, as Babelon, Bernard and Smirnov (infinite volume sine-Grodon model), Smirnov (finite volume quantum KdV model), Lukyanov (finite volume sinh-Gordon model), Derkachov, Korchemsky and Manashov (non-compact XXX chain), Bytsko and Teschner (lattice sinh-Gordon model) ....
- Need for a systematic development and generalization of the SOV method ....

Plan of the seminar:

- Introduction to the quantum analog concept of Separation of Variables (SOV)[1] in the framework of the Quantum Inverse Scattering Method[2].
- General description of Quantum Separation of Variables for integrable quantum models associated to representations of the Yang-Baxter & Reflection algebras of 6-vertex type.
- Universal characterization of spectrum & "dynamics" of integrable quantum models by SOV.
- Description of the SOV results for XXX open spin 1/2 quantum chains with the most general integrable boundary conditions:
  - Completeness of spectrum description by functional equation of Baxter's type.
  - Matrix elements of local operators, first fundamental results toward the model dynamics.
- Projects.

[1] E. K. Sklyanin, Lect. Notes Phys. 226 (1985) 196.
[2] L.D. Faddeev, E.K. Sklyanin and L.A. Takhtajan, Teor. Mat. Fiz. 40 (1979) 194.

- **Quantum integrability:** (H Hamiltonian, $\mathcal{H}$ quantum space of model)
  $\exists$ $\mathsf{T}(\lambda) \in \mathsf{End}(\mathcal{H})$: i) $[\mathsf{T}(\lambda),\mathsf{T}(\lambda')]=0 \quad \forall\lambda,\lambda'$, ii) $[\mathsf{T}(\lambda),H]=0 \quad \forall\lambda\in\mathbb{C}$,

- Let $\mathsf{Y}_n \in \mathsf{End}(\mathcal{H})$ and $P_n \in \mathsf{End}(\mathcal{H})$ be N couples of canonical conjugate operators:

$$[\mathsf{Y}_n,\mathsf{Y}_m]=[P_n,P_m]=0, \qquad [\mathsf{Y}_n,P_m]=\delta_{n,m}/2\pi i \;\; \forall(n,m)\in\{1,..,\mathsf{N}\}^2,$$

  where $\{\mathsf{Y}_1,\dots,\mathsf{Y}_\mathsf{N}\}$ are simultaneous diagonalizable operators with simple spectrum.

- **SOV Definition**: The Y_n are quantum separate variables for $\mathsf{T}(\lambda)$ if and only if for any T-eigenvalue $t(\lambda)$ and T-eigenstate $|t\rangle$ it holds:

$$|t\rangle = \sum_{\text{over spectrum of } \{\mathsf{Y}_n\}} \prod_{n=1}^{\mathsf{N}} \mathsf{Q}_t^{(n)}(y_n)\; |y_1,\dots,y_\mathsf{N}\rangle,$$

  where the $t(\lambda)$ and the $Q_t^{(n)}(\lambda)$ are solutions of separate equations in y_n of type

$$F_n(y_n,\frac{i}{2\pi}\frac{d}{dy_n},t(y_n))\,Q_t^{(n)}(y_n)=0, \quad \text{for all the } n\in\{1,\dots,\mathsf{N}\}.$$

- The N quantum separate relations are the natural quantum analogue of the classical ones in the Hamilton-Jacobi's approach.

- The Hydrogen atom Hamiltonian represents one natural example of integrable quantum system to which quantum SOV applies, here the y_n are the spherical coordinates r,θ,ϕ.

Quantum description for the Hydrogen atom: integrability and separate variables

By choosing spherical coordinates in the position representation the Hamiltonian reads:

$$\langle r,\theta,\varphi|H=[-(\partial^2/\partial r^2)(\hbar^2 r/2m)+\mathbf{L}^2/2mr^2-e^2/r]\langle r,\theta,\varphi|,$$

where $\mathbf{L}$ is the angular momentum, vector differential operator in θ and φ only.

This quantum system is integrable as $H_3=H$, $H_2=\mathbf{L}^2$ and $H_1=\mathbf{L}_z$ form a C.S.C.O. of conserved charges, $[H,\mathbf{L}^2]=[H,\mathbf{L}_z]=[\mathbf{L}^2,\mathbf{L}_z]=0$ and the separate relations read:

$$F_n(y_n,\frac{i}{2\pi}\frac{\partial}{\partial y_n},h_3(k,l),h_2(l),h_1(m))\Psi_{k,l,m}(r,\theta,\varphi)=0,\quad y_3=r,y_2=\theta,y_1=\varphi.$$

where $h_3(k,l)=E_I/(k+l)^2$, $h_2(l)=l(l+1)\hbar^2$, $h_1(m)=m\hbar$ and:

$$F_3(r,\frac{i}{2\pi}\frac{\partial}{\partial r},h_3(k,l),h_2(l))\equiv-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}r+\frac{h_2(l)}{2mr^2}-\frac{e^2}{r}-h_3(k,l),$$

$$F_2(\theta,\frac{i}{2\pi}\frac{\partial}{\partial\theta},h_2(l),h_1(m))\equiv-\hbar^2\frac{\partial^2}{\partial\theta^2}-\frac{\hbar^2}{\tan\theta}\frac{\partial}{\partial\theta}+\frac{h_1^2(m)}{\sin^2\theta}-h_2(l),$$

$$F_1(\varphi,\frac{i}{2\pi}\frac{\partial}{\partial\varphi},h_1(m))\equiv-i\hbar\frac{\partial}{\partial\varphi}-h_1(m),$$

and the wavefunctions are separated in the eigenvalues of the separate operators (r, θ and φ):

$$\Psi_{k,l,m}(r,\theta,\varphi)\equiv\langle r,\theta,\varphi|\Psi_{k,l,m}\rangle=R_{k,l}(r)Y_l^m(\theta,\varphi)\ \text{ with }\ Y_l^m(\theta,\varphi)=F_l^m(\theta)e^{im\varphi},$$

where $Y_{l,m}(\theta,\varphi)$ are the spherical harmonic functions.

- **Framework: Quantum Inverse Scattering Method (QISM)**
  - Quantum Integrable Model: There exists a one-parameter family of commuting conserved charges $\mathsf{T}(\lambda)$ (Transfer Matrix).
  - $\mathsf{T}(\lambda)$ is written in terms of the generators of the Yang-Baxter or Reflection Algebras associated to representations of some quantum groups.

- **How to define the Quantum Separate Variables?**

  The main idea introduced by Sklyanin is to use the Yang-Baxter algebra commutation relations to identify a set of quantum separate variables Y_n for the Transfer Matrix $\mathsf{T}(\lambda)$.

- **How to implement SOV systematically for integrable quantum models?**

  This is still an open problem for some important classes of integrable quantum models. Our research activity has enlarged the domain of applicability of the SOV approach and allows to state the following:

- **Motivations to use Quantum Separation of Variables**

  The SOV method allows to solve the problems which appear in other more traditional methods, like Bethe ansatz and Baxter's Q-operator, giving:
  a) the proof of completeness of the spectrum description,
  b) the analysis of a larger class of integrable quantum models,
  c) more symmetrical approach to classical and quantum integrability.
  d) universal characterization of spectrum & dynamics of integrable quantum models.

- **Characterization of integrability by quantum inverse scattering method (QISM)**

  Integrable quantum model with Hamiltonian $H \in \text{End}(\mathcal{H})$ and quantum space $\mathcal{H} \equiv \otimes_{n=1}^{\mathsf{N}} \mathcal{H}_{\mathsf{n}}$ (tensor product of local quantum spaces $\mathcal{H}_{\mathsf{n}}$) is characterized by the monodromy matrix $\mathsf{M}_a(\lambda) \in \text{End}(\mathbb{C}^n \otimes \mathcal{H})$, whose matrix elements are operators on $\mathcal{H}$:

$$R_{ab}(\lambda-\mu)\,\mathsf{M}_a(\lambda)\mathsf{M}_b(\mu) = \mathsf{M}_b(\mu)\mathsf{M}_a(\lambda)\,R_{ab}(\lambda-\mu)$$

Yang-Baxter equation

$$\mathsf{T}(\lambda) = \text{Tr}\,\mathsf{M}(\lambda),\ \ [H,\mathsf{T}(\lambda)] = [\mathsf{T}(\mu),\mathsf{T}(\lambda)] = 0.$$

Transfer matrix

  The elements $(\mathsf{M}_a(\lambda))_{i,j} \in \text{End}(\mathcal{H})$ are generators of the Yang-Baxter algebra associated to given representations of quantum groups $U_q(sl(n))$.

  In the case $n=2$, $\mathsf{M}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$ with $A(\lambda), B(\lambda), C(\lambda)$ and $D(\lambda) \in \text{End}(\mathcal{H})$, here the quantum deformation of the determinant:

$$\det{}_q \mathsf{M}(\lambda) = A(\lambda)\,D(\lambda/q) - B(\lambda)\,C(\lambda/q)$$

  is the so-called quantum determinant, a central element of the Yang-Baxter algebra associated to $U_q(sl(2))$.

- **Sklyanin's approach for SOV characterization** If $B(\lambda)$ is diagonalizable and with simple spectrum then the quantum separate variables Y_n for the spectral problem of the transfer matrix $\mathsf{T}(\lambda)$ are defined by the zeros operators of $B(\lambda)$.

Introduction to SOV & SOV characterization of T-spectrum: "Yang-Baxter case"

- SOV-representation is defined in the basis formed by the B-eigenstates $\{\langle \mathbf{y}| \equiv \langle y_1, \ldots, y_\mathsf{N}|\}$ parametrized by the zeros of $B(\lambda)$:

$$\langle \mathbf{y}| \, B(\lambda) = \, b_\mathbf{y}(\lambda) \, \langle \mathbf{y}| \, , \qquad b_\mathbf{y}(\lambda) \, \equiv \, b_0 \prod_{n=1}^{\mathsf{N}} (\lambda/y_n - y_n/\lambda) \; .$$

By the Yang-Baxter commutations relations, it holds:

$$\langle y_1, \ldots, y_k, \ldots, y_N | A(y_k) = a(y_k) \langle y_1, \ldots, y_k/q, \ldots, y_\mathsf{N}|,$$

$$\langle y_1, \ldots, y_k, \ldots, y_N | D(y_k) = d(y_k) \langle y_1, \ldots, y_k q, \ldots, y_\mathsf{N}|,$$

$$d(\lambda/q) a(\lambda) = \det{}_q M(\lambda).$$

- Eigenvalues $t(\lambda)$ and wave functions $\Psi_t(y_1, \ldots., y_\mathsf{N}) \equiv \langle y_1, \ldots, y_\mathsf{N}|t\rangle$ are characterized by:

$$a(y_k)\Psi_t(y_1, .., y_k q^{-1}, .., y_\mathsf{N}) + d(y_k)\Psi_t(y_1, .., y_k q, .., y_\mathsf{N}) = t(y_k)\Psi_t(y_1, .., y_\mathsf{N}).$$

These equations follow by computing the matrix elements $\langle y_1, \ldots., y_\mathsf{N}|\mathsf{T}(y_k)|t\rangle$ and lead to the factorized form:

$$\Psi_t(y_1, \ldots., y_\mathsf{N}) = \prod_{j=1}^{\mathsf{N}} Q_t(y_j),$$

where $Q_t(\lambda)$ is a solution of the Baxter equation.

Derivation of SOV-representations of $A(\lambda)$ and $D(\lambda)$ in the Yang-Baxter case

a) Yang-Baxter commutation relation for the 6-vertex R-matrix:

$$A(\mu)B(\lambda)=\frac{q\lambda/\mu-q^{-1}\mu/\lambda}{\lambda/\mu-\mu/\lambda}B(\lambda)A(\mu)+\frac{q^{-1}-q}{\lambda/\mu-\mu/\lambda}B(\mu)A(\lambda)$$

b) The centrality of the quantum determinant:

$$\det_q M(\lambda)\equiv A(\lambda)D(\lambda/q)-B(\lambda)C(\lambda/q)=a(\lambda)d(\lambda/q).$$

- Action of a) for $\mu=y_k$ on the B-eigenstate $\langle y_1,..,y_{\mathsf{N}}|$:

$$(\langle y_1,\dots,y_k,\dots,y_{\mathsf{N}}|A(y_k))\,B(\lambda)=b_{y_1,..,y_k/q,..,y_{\mathsf{N}}}(\lambda)\langle y_1,\dots,y_k,\dots,y_N|A(y_k)$$

where: $b_{y_1,..,y_k/q,..,y_{\mathsf{N}}}(\lambda)\equiv\frac{(q\lambda/y_k-q^{-1}y_k/\lambda)}{(\lambda/y_k-y_k/\lambda)}b_{y_1,..,y_k,..,y_{\mathsf{N}}}(\lambda).$

- Simplicity of $B(\lambda)\ \rightarrow\ \langle y_1,\dots,y_k,\dots,y_{\mathsf{N}}|A(y_k)\propto\langle y_1,\dots,y_k/q,\dots,y_{\mathsf{N}}|$.

- Yang-Baxter algebra $\&$ quantum determinant imply:

$$\begin{aligned}\langle y_1,\dots,y_k,\dots,y_N|A(y_k)&=a(y_k)\langle y_1,\dots,y_k/q,\dots,y_{\mathsf{N}}|,\\ \langle y_1,\dots,y_k/q,\dots,y_N|D(y_k/q)&=d(y_k/q)\langle y_1,\dots,y_k,\dots,y_{\mathsf{N}}|.\end{aligned}$$

- **Quantum Inverse Scattering Formulation of Open Integrable Quantum Models:**

The boundary monodromy matrix $\mathcal{U}_-(\lambda)$ satisfy the following Reflection algebra[3]:

$$R_{12}(\lambda/\mu)\,\mathcal{U}_{-,1}(\lambda)\,R_{12}(\lambda\mu)\,\mathcal{U}_{-,2}(\mu)=\mathcal{U}_{-,2}(\mu)\,R_{12}(\lambda\mu)\,\mathcal{U}_{-,1}(\lambda)\,R_{12}(\lambda/\mu),$$

where R is for example the 6-vertex trigonometric R-matrix and $\mathcal{U}_-(\lambda)$ is defined by[4]:

$$\mathcal{U}_-(\lambda)=M_0(\lambda)K_-(\lambda)\hat{M}_0(\lambda)=\left(\begin{array}{cc}\mathcal{A}_-(\lambda) & \mathcal{B}_-(\lambda)\\ \mathcal{C}_-(\lambda) & \mathcal{D}_-(\lambda)\end{array}\right),\ \hat{M}(\lambda)=(-1)^{\mathsf{N}}\,\sigma_0^y\,M^{t_0}(1/\lambda)\,\sigma_0^y$$

where $M_0(\lambda)$ is a solution to the Yang-Baxter equation and

$$K_\pm(\lambda)\equiv\left(\begin{array}{cc}a_\pm(\lambda) & b_\pm(\lambda)\\ c_\pm(\lambda) & d_\pm(\lambda)\end{array}\right)$$

a scalar solution of the reflection algebra, then the transfer matrix[9]:

$$\mathsf{T}(\lambda)\equiv\mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}=a_+(\lambda)\,\mathcal{A}_-(\lambda)+d_+(\lambda)\,\mathcal{D}_-(\lambda)+c_+(\lambda)\,\mathcal{B}_-(\lambda)+b_+(\lambda)\,\mathcal{C}_-(\lambda)$$

defines a one parameter family of conserved charges for a class of integrable quantum models.

[3]Cherednik I V, 1984 Theor. Math. Phys. 61 977

[4]Sklyanin E K, 1988 J. Phys. A: Math. Gen. 21 2375

Introduction to SOV & SOV characterization of T-spectrum: "Reflection algebra case"

- SOV-representation is defined in the basis formed by the $\mathcal{B}_-$-eigenstates $\{\langle \mathbf{y}| \equiv \langle y_1, \ldots, y_{\mathsf{N}}|\}$ parametrized by the zeros of $\mathcal{B}_-(\lambda)$:

$$\langle \mathbf{y}|\, \mathcal{B}_-(\lambda) = b_{-,\mathbf{y}}(\lambda)\, \langle \mathbf{y}|, \quad b_{-,\mathbf{y}}(\lambda) \equiv b_-(\lambda) \prod_{n=1}^{\mathsf{N}} (\lambda/y_n - y_n/\lambda) \prod_{n=1}^{\mathsf{N}} (\lambda y_n - 1/(y_n\lambda))\,.$$

By the reflection algebra commutations relations, it holds:

$$\langle y_1, \ldots, y_k, \ldots, y_N|\mathcal{A}_-(y_k^{\pm 1}) = \mathsf{a}_-(y_k^{\pm 1})\langle y_1, \ldots, y_k q^{\mp 1}, \ldots, y_{\mathsf{N}}|,$$

$$\mathcal{D}_-(\lambda) = \left[(\lambda^2/q - q/\lambda^2)\mathcal{A}_-(1/\lambda) + (\lambda^2/q - q/\lambda^2)\mathcal{A}_-(\lambda)\right] / (\lambda^2 - 1/\lambda^2)$$

$$\frac{\det_q \mathcal{U}_-(\lambda)}{(\lambda^2/q - q/\lambda^2)} = \mathcal{A}_-(\lambda q^{1/2})\mathcal{A}_-(q^{1/2}/\lambda) + \mathcal{B}_-(\lambda q^{1/2})\mathcal{C}_-(q^{1/2}/\lambda) = \mathsf{a}_-(\lambda q^{1/2})\mathsf{a}_-(q^{1/2}/\lambda).$$

- In the triangular case $(b_+(\lambda) = 0)$ the transfer matrix reads:

$$\mathsf{T}(\lambda) \equiv \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\} = \bar{a}_+(\lambda)\, \mathcal{A}_-(\lambda) + \bar{a}_+(1/\lambda)\, \mathcal{A}_-(1/\lambda) + c_+(\lambda)\, \mathcal{B}_-(\lambda).$$

defined $a(\lambda) \equiv \bar{a}_+(\lambda)\mathsf{a}_-(\lambda)$ then the eigenvalues and eigenstates are characterized by:

$$a(y_k)\Psi_t(y_1, .., y_k q^{-1}, .., y_{\mathsf{N}}) + a(1/y_k)\Psi_t(y_1, .., y_k q, .., y_{\mathsf{N}}) = t(y_k)\Psi_t(y_1, .., y_{\mathsf{N}}).$$

These equations lead to factorized wavefunctions by Baxter equation solutions $Q_t(y_j)$:

$$\Psi_t(y_1, \ldots., y_{\mathsf{N}}) = \prod_{j=1}^{N} Q_t(y_j).$$

- **Some general statement about SOV representation: Summary**

a) The quantum separate variables for $\mathsf{T}(\lambda) = \mathrm{tr}_0\{M(\lambda)\}$, associated to $\mathsf{M}(\lambda)$ solution of 6-vertex type Yang-Baxter equations, are the operators zero Y_n of the $B(\lambda)$, if it is diagonalizable and with simple spectrum.

b) The quantum separate variables for $\mathsf{T}(\lambda) \equiv \mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}$, (with $K_+(\lambda)$ triangular), associated to $\mathcal{U}_-(\lambda)$ solution of 6-vertex type Reflection equations, are the operators zero Y_n of the generator $\mathcal{B}_-(\lambda)$ if it is diagonalizable and with simple spectrum.

$\mathsf{T}(\lambda)\,|t\rangle = t(\lambda)|t\rangle$, $|t\rangle$ **eigenvector of** $\mathsf{T}(\lambda)$, $t(\lambda)$ **eigenvalue of** $\mathsf{T}(\lambda)$,

$\Updownarrow$

SOV representation: $|t\rangle = \sum_{\{y\}} \prod_{j=1}^{N} Q_t(y_j)|y_1, ..., y_N\rangle, \quad Q_t(y_j) \in \mathbb{C},$

Baxter's equation: $t(y_j)Q_t(y_j) = a(y_j)Q_t(y_j/q) + d(y_j)Q_t(y_j q), \quad q \in \mathbb{C}.$

Baxter's equations in separate variables are implied by the Yang-Baxter or Reflection equations.

- **Important classes of models require a generalization of these statements on SOV:**

Examples for which generalizations are required and we have found them are:

i) Integrable quantum models associated to dynamical 6-vertex and 8-vertex Yang-Baxter algebras.

ii) Integrable quantum models associated to 6-vertex/8-vertex Reflection algebra for the most general integrable boundary conditions.

Toward quantum dynamics: definition and problems to solve

Definition Form factors $\langle t'|\mathcal{O}_n|t\rangle$ are the matrix elements of a local operator $\mathcal{O}_n$ between the eigencovector $\langle t'|$ and the eigenvector $|t\rangle$ of $\mathsf{T}(\lambda)$.

The form factors are the "elementary objects" w.r.t. any time dependent correlation function can be expanded by using the decomposition of the identity in the transfer matrix eigenbasis:

$$\langle t'|\mathcal{O}_n(\theta_1)\mathcal{O}_m(\theta_2)|t''\rangle = \sum_{t\in\Sigma_{\mathsf{T}}} \frac{\langle t'|\mathcal{O}_n|t\rangle\langle t|\mathcal{O}_m|t''\rangle}{\langle t|t\rangle} e^{(h_{t'}-h_t)\theta_1+(h_t-h_{t''})\theta_2}, \forall n<m \in \{1,\ldots,\mathsf{N}\}$$

where $h_{t'}$ and $h_{t''}$ are the Hamiltonian eigenvalues on the eigenstates $|t''\rangle$ and $|t'\rangle$ and by definition of time evolution operator, it holds $\mathcal{O}_n(\theta) \equiv e^{iH\theta}\mathcal{O}_n e^{-iH\theta}$.

Two difficult problems to solve:

i) Reconstruction of local operators $\mathcal{O}_n$ in terms of quantum SOV variables and their conjugates.

$\hookrightarrow$ Algebraic computation of the action of local operators $\mathcal{O}$ on the eigenvector $|t\rangle$.

ii) Scalar product $\langle\alpha|t\rangle$ under the form of determinant, where $\langle\alpha|$ is a generic separate state.

Steps i) and ii) allow us to get in a determinant form the form factors of a basis of operators.

First results toward quantum dynamics by SOV:

Universal characterization of the form factors by SOV-method[5]**:** For all the integrable models associated to finite dimensional quantum space whose transfer matrix spectrum (eigenvalues and eigenstates) has been characterized by SOV method the following statements hold. There exists a basis $\mathbb{B}_{\mathcal{H}}$ in End$(\mathcal{H})$ such that for any $\mathsf{O} \in \mathbb{B}_{\mathcal{H}}$ the matrix elements on the transfer matrix eigenstates read:

$$\langle t'|\mathsf{O}|t\rangle = \det_{\mathsf{N}} ||\Phi_{a,b}^{(\mathsf{O},t,t)}||, \quad \Phi_{a,b}^{(\mathsf{O},t,t)} \equiv \sum_{c=1}^{p} F_{\mathsf{O},b}(y_a^{(c)}) Q_t(y_a^{(c)}) Q_{t'}(y_a^{(c)}) (y_a^{(c)})^{2(b-1)}.$$

The coefficients $F_{\mathsf{O},b}(y_a^{(c)})$ characterize the operator O and they are computed algebraically thanks to the reconstruction by the quantum separate variables of O.

In the case of XXX and XXZ spin 1/2 chains with closed[6] or open integrable boundaries[7], these scalar products and matrix elements of local operators admit rewriting in terms of formulae of Izergin's and Slavnov's type.

This represents an important step to apply known techniques allowing for the analysis of the thermodynamic limit to integrable quantum models previously not analyzable by other methods.

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[5] Grosjean, Maillet, Niccoli (2012) and subsequent papers.

[6] In the closed XXX chain these formulae are reminiscent of those first obtained by Kostov (2012) by a different approach.

[7] Kitanine, Maillet, Niccoli, Terras (2015-2017)

The XXX quantum spin-1/2 chain with the most general integrable boundaries

• The local Hamiltonian, the rational 6-vertex monodromy matrix and R-matrix:

$$H=\sum_{i=1}^{N-1}\left[\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\sigma_i^z\sigma_{i+1}^z\right]+\frac{\eta}{\zeta_-}\left[\sigma_1^z+2\kappa_-\left(e^{\tau_-}\sigma_1^+ + e^{-\tau_-}\sigma_1^-\right)\right]+\frac{\eta}{\zeta_+}\left[\sigma_N^z+2\kappa_+\left(e^{\tau_+}\sigma_N^+ + e^{-\tau_+}\sigma_N^-\right)\right]$$

and $\mathsf{M}_0(\lambda)=R_{0\mathsf{N}}(\lambda-\xi_\mathsf{N})\cdots R_{02}(\lambda-\xi_2)\,R_{01}(\lambda-\xi_1)$, with $R(\lambda)=\begin{pmatrix}\lambda+\eta & 0 & 0 & 0\\ 0 & \lambda & \eta & 0\\ 0 & \eta & \lambda & 0\\ 0 & 0 & 0 & \lambda+\eta\end{pmatrix}$.

The σ_i^α are the Pauli matrices on the local quantum space $\mathcal{H}_i\simeq\mathbb{C}^2$ at site i and $\zeta_\pm$, $\kappa_\pm$ and $\tau_\pm$ are the boundary parameters.

• The transfer matrix $\mathsf{T}(\lambda)\equiv\mathrm{tr}_0\{K_+(\lambda)\mathcal{U}_-(\lambda)\}$, associated to the triangular boundaries

$$K_+(\lambda)=\mathrm{I}+\frac{\lambda+\eta/2}{\bar\zeta_+}(\sigma^z+\bar{\mathsf{c}}_+\sigma^-),\qquad K_-(\lambda)=\mathrm{I}+\frac{\lambda-\eta/2}{\bar\zeta_-}(\sigma^z+\bar{\mathsf{b}}_-\sigma^+),$$

with $\bar\zeta_\pm$, $\bar{\mathsf{c}}_+$, $\bar{\mathsf{b}}_-$ fixed by the original boundaries parameters by $\bar\zeta_\pm=\zeta_\pm/\sqrt{1+4\kappa_\pm^2}$,

$$\bar{\mathsf{c}}_+=\frac{2\kappa_+e^{-\tau_+}\left[1+\frac{(1+\sqrt{1+4\kappa_+^2})(1-\sqrt{1+4\kappa_-^2})}{4\kappa_+\kappa_-e^{\tau_--\tau_+}}\right]}{\sqrt{1+4\kappa_+^2}},\quad \bar{\mathsf{b}}_-=\frac{2\kappa_-e^{\tau_-}\left[1+\frac{(1-\sqrt{1+4\kappa_+^2})(1+\sqrt{1+4\kappa_-^2})}{4\kappa_+\kappa_-e^{\tau_--\tau_+}}\right]}{\sqrt{1+4\kappa_-^2}},$$

defines this local Hamiltonian by the similarity matrix $\Gamma_W\equiv\otimes_{n=1}^N W_n$ as it follows :

$$H=\frac{1}{2\,\eta^{2N-1}}\,\Gamma_W^{-1}\frac{d}{d\lambda}\mathsf{T}(\lambda)_{|\lambda=\eta/2,\xi_i=0}\Gamma_W\,,\quad W_n\equiv\mathrm{I}_n-\frac{1-\sqrt{1+4\kappa_+^2}}{2\kappa_+e^{-\tau_+}}\,\sigma_n^+ +\frac{1-\sqrt{1+4\kappa_-^2}}{2\kappa_-e^{\tau_-}}\,\sigma_n^-\,.$$

SOV characterization of the transfer matrix spectrum by discrete system of equations

Theorem 1. *SOV applies for generic values of inhomogeneities and* $\bar{\mathsf{b}}_- \neq 0$*, the spectrum* $\Sigma_{\mathcal{T}}$ *of* $\mathcal{T}(\lambda)$ *is simple and coincides with the set of solutions to the discrete system of equations*

$$t(\xi_n^{(0)})\, t(\xi_n^{(1)}) = \mathsf{A}_{\bar{\zeta}_+,\bar{\zeta}_-}(\xi_n^{(0)})\, \mathsf{A}_{\bar{\zeta}_+,\bar{\zeta}_-}(-\xi_n^{(1)}), \;\; 1 \leqslant n \leqslant N,$$

for the functions $t(\lambda) = \frac{2+\bar{\mathsf{b}}_-\bar{\mathsf{c}}_+}{\bar{\zeta}_+\,\bar{\zeta}_-}(\lambda^2 - (\eta/2)^2) \prod_{b=1}^{N} (\lambda^2 - t_b^2) + 2(-1)^N a(0)\, d(-\eta)$*,* *where* $\xi_n^{(h)} = \xi_n + \eta/2 - h\eta$ *and we have defined:*

$$\mathsf{A}_{\bar{\zeta}_+,\bar{\zeta}_-}(\lambda) \equiv (-1)^N \frac{2\lambda+\eta}{2\lambda}\, \frac{(\lambda - \frac{\eta}{2} + \bar{\zeta}_+)(\lambda - \frac{\eta}{2} + \bar{\zeta}_-) a(\lambda)\, d(-\lambda)}{\bar{\zeta}_+\,\bar{\zeta}_-}, \;\; d(\lambda+\eta) \equiv a(\lambda) \equiv \prod_{n=1}^{N} (\lambda - \xi_n + \eta/2).$$

The right and left $\mathcal{T}(\lambda)$*-eigenstates associated to the eigenvalue* $t(\lambda)$ *are defined by*

$$\langle \mathbf{h}|t\rangle = \prod_{n=1}^{N} Q_t(\xi_n^{(h_n)}), \;\; \langle t|\mathbf{h}\rangle = \prod_{n=1}^{N} (f_n\, g_n)^{h_n}\, Q_t(\xi_n^{(h_n)}), \quad \frac{Q_t(\xi_n^{(1)})}{Q_t(\xi_n^{(0)})} = \frac{t(\xi_n^{(0)})}{\mathsf{A}_{\bar{\zeta}_+,\bar{\zeta}_-}(\xi_n^{(0)})},$$

where $g_n \equiv \frac{(\xi_n+\bar{\zeta}_+)(\xi_n+\bar{\zeta}_-)}{(\xi_n-\bar{\zeta}_+)(\xi_n-\bar{\zeta}_-)}$*, and* $f_n \equiv \prod_{a=1\, a\neq n}^{N} \frac{(\xi_n-\xi_a+\eta)(\xi_n+\xi_a+\eta)}{(\xi_a-\xi_n+\eta)(\xi_a+\xi_n-\eta)}$*, in the SoV basis:*

$$\begin{aligned} \langle \mathbf{h}|\mathcal{B}_-(\lambda) &= \mathsf{b}_{-,\mathbf{h}}(\lambda)\langle \mathbf{h}|, \\ \mathcal{B}_-(\lambda)|\mathbf{h}\rangle &= \mathsf{b}_{-,\mathbf{h}}(\lambda)|\mathbf{h}\rangle, \end{aligned} \quad \text{where} \quad \mathsf{b}_{-,\mathbf{h}}(\lambda) = \bar{\mathsf{b}}_- \frac{\lambda - \eta/2}{\bar{\zeta}_-} \prod_{n=1}^{N} (\lambda^2 - (\xi_n^{(h_n)})^2).$$

where $\langle \mathbf{k}|\mathbf{h}\rangle = \delta_{\mathbf{h},\mathbf{k}}\, N_{\boldsymbol{\xi},-}/\widehat{V}(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)})$ *and* $\widehat{V}(x_1, \ldots, x_N) = \prod_{j<k} (x_k^2 - x_j^2)$*.*

SOV characterization of the transfer matrix spectrum by TQ functional equations

Theorem 2. *a) For generic inhomogeneities and* $\bar{\mathsf{b}}_- \neq 0$*, then* $t(\lambda) \in \Sigma_{\mathcal{T}}$ *iff exists a unique polynomial* $Q_t(\lambda) = \prod_{b=1}^{q} \left(\lambda^2 - \lambda_b^2\right)$*, with* $\lambda_1, \ldots, \lambda_q \in \mathbb{C} \setminus \{ \pm \xi_1^{(0)}, \ldots, \pm \xi_N^{(0)} \}$*, such that*

$$t(\lambda)\, Q_t(\lambda) = \mathsf{A}_{\bar{\zeta}_+,\bar{\zeta}_-}(\lambda)\, Q_t(\lambda - \eta) + \mathsf{A}_{\bar{\zeta}_+,\bar{\zeta}_-}(-\lambda)\, Q_t(\lambda + \eta) + F(\lambda),$$

$F(\lambda) = \frac{\bar{\mathsf{b}}_- \bar{\mathsf{c}}_+}{\bar{\zeta}_- \bar{\zeta}_+} (\lambda^2 - (\eta/2)^2) \prod_{b=1}^{N} \prod_{h=0}^{1} (\lambda^2 - (\xi_b^{(h)})^2)$ *,* $q = N$ *for* $\bar{\mathsf{c}}_+ \neq 0$ *and* $q \leq N$ *for* $\bar{\mathsf{c}}_+ = 0$.

b) Let us define the following states in the SoV basis:

$$|\omega_R\rangle = \sum_{\mathbf{h} \in \{0,1\}^N} \frac{\widehat{V}\left(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)}\right) |\mathbf{h}\rangle}{N_{\boldsymbol{\xi},-}}, \quad \langle \omega_L| = \sum_{\mathbf{h} \in \{0,1\}^N} \frac{\widehat{V}\left(\xi_1^{(h_1)}, \ldots, \xi_N^{(h_N)}\right) \prod_{a=1}^{N} (f_a\, g_a)^{h_a} \langle \mathbf{h}|}{N_{\boldsymbol{\xi},-}},$$

then the transfer matrix eigenstates admit the following algebraic Bethe ansatz form $|t\rangle = \prod_{a=1}^{q} \mathcal{B}(\lambda_a) |\omega_R\rangle$ *and* $\langle t| = \langle \omega_L| \prod_{a=1}^{q} \mathcal{B}(\lambda_a)$*, where* $\mathcal{B}(\lambda) \equiv \frac{(-1)^N \bar{\zeta}_- \mathcal{B}_-(\lambda)}{(\lambda - \eta/2) \bar{\mathsf{b}}_-}$.

Definition: Right $|\alpha\rangle$ and left $\langle \beta|$ separate states are defined by the following factorized form in the SoV basis $\langle \mathbf{h}|\alpha\rangle = \prod_{n=1}^{N} \alpha_n^{(h_n)}$, $\langle \beta|\mathbf{h}\,\rangle = \prod_{n=1}^{N} (f_n\, g_n)^{h_n} \beta_n^{(h_n)}$.

Remark: Let $\alpha(\lambda) = \prod_{k=1}^{n_\alpha} (\lambda^2 - \alpha_k^2)$ and $\beta(\lambda) = \prod_{k=1}^{n_\beta} (\lambda^2 - \beta_k^2)$ be two polynomials such that $\alpha(\xi_n^{(h_n)}) = \alpha_n^{(h_n)}$ and $\beta(\xi_n^{(h_n)}) = \beta_n^{(h_n)}$ then separate states admit the following representations:

$$|\alpha\rangle = \prod_{a=1}^{n_\alpha} \mathcal{B}(\alpha_a) |\omega_R\rangle, \quad \langle \beta| = \langle \omega_L| \prod_{a=1}^{n_\beta} \mathcal{B}(\beta_a)$$

Scalar product of separate states: first SOV representation

For a set of arbitrary variables $\{x\} \equiv \{x_1, \dots, x_L\}$ and a function f, we define the function

$$\mathcal{A}_{\{x\}}[f] = \det_{1\leqslant i,j\leqslant L}\left[\sum_{\epsilon\in\{+,-\}} f(\epsilon x_i)\left(x_i+\epsilon\frac{\eta}{2}\right)^{2(j-1)}\right]\det^{-1}_{1\leqslant i,j\leqslant L}\left[x_i^{2(j-1)}\right],$$

and

$$f_{\xi_+,\xi_-,\{z\}}(\lambda) = \lambda^{-1}(\lambda+\xi_+)(\lambda+\xi_-)\prod_{m=1}^{M}(\lambda^2-z_m^2)/((\lambda+\frac{\eta}{2})^2-z_m^2),$$

for ξ_+, ξ_- and $\{z\} \equiv \{z_1, \dots, z_M\}$ a set of arbitrary variables.

Proposition 1. *Let* $\alpha(\lambda) = \prod_{k=1}^{n_\alpha}(\lambda^2-\alpha_k^2)$ *and* $\beta(\lambda) = \prod_{k=1}^{n_\beta}(\lambda^2-\beta_k^2)$ *be two polynomials in* λ^2 *of respective degree* n_α *and* n_β*, then the scalar products of the associated separate states,*

$$\langle\alpha\,|\,\beta\rangle = \langle\beta\,|\,\alpha\rangle = \langle\omega_L|\prod_{k=1}^{n_\alpha}\mathcal{B}(\alpha_k)\prod_{k=1}^{n_\beta}\mathcal{B}(\beta_k)|\omega_R\rangle,$$

admit the following determinant representations:

$$\langle\alpha\,|\,\beta\rangle = N_{\alpha,\beta,\bar\zeta_+}\,\mathcal{A}_{\{\xi\}}\left[f_{\bar\zeta_+,\bar\zeta_-,\{\alpha\}\cup\{\beta\}}\right],$$

where we have defined

$$N_{\alpha,\beta,\bar\zeta_+} \equiv (-1)^N\prod_{n=1}^{N}\frac{\alpha(\xi_n^{(0)})\,\beta(\xi_n^{(0)})\,\alpha(\xi_n^{(1)})\,\beta(\xi_n^{(1)})}{(\xi_n-\bar\zeta_+)\,\bar{\mathsf{b}}_-\,\alpha(\xi_n)\,\beta(\xi_n)}$$

Main determinant identities to rewrite scalar product of separate states

Identity 1. *Let* $\{x\} \equiv \{x_1, \dots, x_L\}$ *and* $\{z\} \equiv \{z_1, \dots, z_M\}$ *be two sets of arbitrary complex numbers and let* $\theta(x) \equiv \{0 \text{ if } x < 0, 1 \text{ if } x \geq 0\}$*. Then, if* ξ_+ *and* ξ_- *are such that* $(\xi_+ + \xi_-)/\eta \notin \{1, \dots, M - L\}$ *(non-empty set only for* $M > L$*), it holds:*

$$\mathcal{A}_{\{x\}}\left[f_{\xi_+,\xi_-,\{z\}}\right] = p_{\xi_++\xi_-,L,M}\ \mathcal{A}_{\{z\}}\left[f_{\frac{\eta}{2}-\xi_+,\frac{\eta}{2}-\xi_-,\{x\}}\right]$$

where $p_{x,L,M} \equiv (-1)^M \prod_{a=0}^{|L-M|-1}(2(x+\eta(a+(L-M)\theta(M-L))))^{1-2\theta(M-L)}$.

For $L \geqslant M$ we define the function $\mathcal{S}_{\{x\},\{y\}}[f] = \frac{\widehat{V}(x_1-\frac{\eta}{2},\dots,x_M-\frac{\eta}{2})}{\widehat{V}(x_1+\frac{\eta}{2},\dots,x_M+\frac{\eta}{2})} \frac{\det_L \mathcal{S}_{\mathbf{x},\mathbf{y}}[f]}{\widehat{V}(x_M,\dots,x_1)\,\widehat{V}(y_1,\dots,y_L)}$,

$$\left[\,\mathcal{S}_{\mathbf{x},\mathbf{y}}[f]\,\right]_{i,k} = \sum_{\epsilon\in\{+,-\}} f(\epsilon y_i) X(y_i+\epsilon\eta) \begin{cases} \frac{f(-x_k)}{(y_i+\epsilon\frac{\eta}{2})^2-(x_k+\frac{\eta}{2})^2} - \frac{f(x_k)\varphi_{\{x\}}(x_k)}{(y_i+\epsilon\frac{\eta}{2})^2-(x_k-\frac{\eta}{2})^2} & \text{if } k \leqslant M, \\ \left(y_i+\epsilon\frac{\eta}{2}\right)^{2(k-M-1)} & \text{if } k > M, \end{cases}$$

$\varphi_{\{x\}}(\lambda) = (2\lambda-\eta)X(\lambda+\eta)/((2\lambda+\eta)X(\lambda-\eta))$ and $X(\lambda) = \prod_{m=1}^{M}(\lambda^2 - x_m^2)$.

Identity 2. *Let us suppose that* $L \geqslant M$*, then for any function* f*,*

$$\mathcal{A}_{\{x\}\cup\{y\}}[f] = \mathcal{S}_{\{x\},\{y\}}[f].$$

Main ideas to prove the determinant identities

The main identities to prove are the following:

$$\mathcal{A}_{\{z\}}[f_{\xi_+,\xi_-,\{x\}}] = \mathcal{I}_{\xi_+,\xi_-}(\{z\},\{x\}) = (-1)^L \mathcal{I}_{\tilde{\xi}_+,\tilde{\xi}_-}(\{x\},\{z\}) = (-1)^L \mathcal{A}_{\{x\}}[f_{\tilde{\xi}_+,\tilde{\xi}_-,\{z\}}].$$

$\tilde{\xi}_\pm \equiv \frac{\eta}{2} - \xi_\pm$ and $M=L$, where we have defined a generalized Izergin's determinant

$$\mathcal{I}_{\xi_+,\xi_-}(\{z\},\{x\}) = \frac{\prod_{j,k=1}^{L}(z_j^2 - x_k^2)\,\det_{1\leqslant i,j\leqslant L}\left[\sum_{\epsilon\in\{+,-\}}\epsilon\frac{(z_i+\epsilon\xi_+)(z_i+\epsilon\xi_-)}{z_i\left[(z_i+\epsilon\frac{\eta}{2})^2-x_j^2\right]}\right]}{\prod_{j<k}(z_j^2 - z_k^2)(x_k^2 - x_j^2)}.$$

The symmetry of the generalized Izergin's determinant follows from the identity:

$$\sum_{\epsilon\in\{+,-\}}\epsilon\frac{(z+\epsilon\xi_+)(z+\epsilon\xi_-)}{z\left[(z_i+\epsilon\frac{\eta}{2})^2 - x^2\right]} = \sum_{\epsilon\in\{+,-\}}\epsilon\frac{(x+\epsilon\tilde{\xi}_+)(x+\epsilon\tilde{\xi}_-)}{x\left[(x+\epsilon\frac{\eta}{2})^2 - z^2\right]}$$

while the identity $\mathcal{A}_{\{z\}}[f_{\xi_+,\xi_-,\{x\}}] = \mathcal{I}_{\xi_+,\xi_-}(\{z\},\{x\})$ is proven multiplying and dividing $\mathcal{A}_{\{z\}}[f_{\xi_+,\xi_-,\{x\}}]$ for $\det_L\left[\mathcal{C}^X\right] = \widehat{V}(x_L,\ldots,x_1)$ and observing that:

$$\sum_{j=1}^{L}\mathcal{C}^X_{j,k}\sum_{\epsilon\in\{+,-\}} f_{\xi_+,\xi_-,\{x\}}(\epsilon z_i)\left(z_i+\epsilon\frac{\eta}{2}\right)^{2(j-1)} = \prod_{\ell=1}^{L}\left(z_i^2 - x_\ell^2\right)\sum_{\epsilon\in\{+,-\}}\epsilon\frac{(z_i+\epsilon\xi_+)(z_i+\epsilon\xi_-)}{z_i\left[(z_i+\epsilon\frac{\eta}{2})^2 - x_k^2\right]},$$

where the $L\times L$ matrix $\mathcal{C}^X$ has elements defined by $\sum_{j=1}^{L}\mathcal{C}^X_{j,k}\,\lambda^{2(j-1)} = \prod_{\substack{\ell=1\\ \ell\neq k}}^{L}(\lambda^2 - x_\ell^2)$.

Scalar product of separate states: new SOV representation

Theorem 3. *Let* $n_\beta \geqslant n_\alpha$*, then the scalar product of the separate states* $\langle\alpha|$ *and* $|\beta\rangle$ *reads:*

$$\langle\alpha|\beta\rangle = N_{\alpha,\beta,\bar{\zeta}_+}\, p_{\bar{\zeta}_+ + \bar{\zeta}_-, N, n_\alpha+n_\beta}\, \mathcal{A}_{\{\alpha\}\cup\{\beta\}}\big[f_{-\bar{\zeta}_+ + \frac{\eta}{2}, -\bar{\zeta}_- + \frac{\eta}{2}, \{\xi\}}\big]$$

or:

$$\langle\alpha|\beta\rangle = N_{\alpha,\beta,\bar{\zeta}_+}\, p_{\bar{\zeta}_+ + \bar{\zeta}_-, N, n_\alpha+n_\beta}\, \mathcal{S}_{\{\alpha\},\{\beta\}}\big[f_{-\bar{\zeta}_+ + \frac{\eta}{2}, -\bar{\zeta}_- + \frac{\eta}{2}, \{\xi\}}\big].$$

Let $\bar{\mathsf{c}}_+ = 0$*, then the scalar product* $\langle t|\beta\rangle = \langle\beta|t\rangle = 0$ *if* $n_\beta < q$*, while for* $n_\beta \geqslant q$ *it reads:*

$$\langle t|\beta\rangle = \frac{(-1)^{q+n_\beta} p_{\bar{\zeta}_+ + \bar{\zeta}_-, N, q+n_\beta}}{\prod_{n=1}^N \big[(\xi_n - \bar{\zeta}_+)\, \bar{\mathsf{b}}_-\big]} \prod_{k=1}^q \frac{(\lambda_k - \frac{\eta}{2} + \bar{\zeta}_+)(\lambda_k - \frac{\eta}{2} + \bar{\zeta}_-)}{\lambda_k}\, a(\lambda_k)\, d(-\lambda_k)$$

$$\times \prod_{i=1}^{n_\beta} \frac{2\bar{\zeta}_+\bar{\zeta}_-\, Q_t(\beta_i)}{\eta^2 - 4\beta_i^2}\, \frac{\widehat{V}(\lambda_1 - \frac{\eta}{2}, \ldots, \lambda_q - \frac{\eta}{2})}{\widehat{V}(\lambda_1 + \frac{\eta}{2}, \ldots, \lambda_q + \frac{\eta}{2})}\, \frac{\det_{n_\beta} \mathcal{S}_t(\{\beta\})}{\widehat{V}(\lambda_q, \ldots, \lambda_1)\, \widehat{V}(\beta_1, \ldots, \beta_{n_\beta})},$$

where

$$\big[\,\mathcal{S}_t(\{\beta\})\big]_{i,k} = \begin{cases} \dfrac{\partial\, t(\beta_i)}{\partial \lambda_k} & \text{if } k \leqslant q, \\[2ex] \displaystyle\sum_{\epsilon\in\{+,-\}} \epsilon\, \mathsf{A}_{\bar{\zeta}_+,\bar{\zeta}_-}(-\epsilon\beta_i)\, \frac{Q_t(\beta_i + \epsilon\eta)}{Q_t(\beta_i)} \left(\beta_i + \epsilon\frac{\eta}{2}\right)^{2(k-q)-1} & \text{if } k > q. \end{cases}$$

Results in the SOV framework:

• A systematic development of the SOV method and new exact results achieved for integrable quantum models like:

○ The lattice sine-Gordon model, the τ_2-model and the chiral Potts model associated to general cyclic representations of the Yang-Baxter and Reflection Algebras.

○ The XYZ quantum spin 1/2 chains and XXZ with arbitrary spin representations and the most general integrable closed/open boundary conditions.

○ The SOS models associated to spin 1/2 representations of the dynamical Yang-Baxter and Reflection Algebras of elliptic and trigonometric type.

• Universal features proven for integrable quantum models associated to representations of the Yang-Baxter and Reflection Algebras of $U_q(\hat{sl}(2))$ and $\mathrm{E}_{\tau,q}(\hat{sl}(2))$ type:

○ Spectrum: the eigenvalues and eigenstates of the Hamiltonian of the model are completely characterized by classifying all the solutions to a given set of equations (Baxter's second order difference equations) in the spectrum of the separate variables,

○ Dynamics: the scalar products for eigenstates and the form factors of local operators admit determinant representations.

Related projects: I am developing with my collaborators two simultaneous lines of research.

I) To complete the exact characterization of the dynamics for the models already analyzed: Computation of correlation functions.

Research Groups:	Projects:
• ENS, Lyon, France: J.-M.Maillet et al.	XXZ spin chains and sine-Gordon model.
• ENS, Lyon, France: J.-M.Maillet et al.	τ_2-model and chiral Potts model with general integrable boundaries.
• LPTM, Orsay, France: V.Terras et al.	Dynamical 6-vertex and elliptic 8-vertex models
• IMB, Dijon, France: N.Kitanine et al.	Open integrable quantum systems and out of equilibrium statistical mechanics: PASEP.

II) Generalization of SOV method for further advanced integrable quantum models:

• ENS, Lyon, France: J.-M.Maillet et al.	Characterization of spectrum and dynamics of spin chains associated to higher rank (super)quantum groups.

These last models should lead to the SOV tools for the solution of models like the Hubbard model of central interest both in Condensed Matter Theory and in Gauge Theory by AdS/CFT.