Multidimensional consistency, Lagrangian multi-forms and quantization

Frank Nijhoff, *University of Leeds*

(joint work with Steven King & Sarah Lobb)

Correlation functions of quantum integrable systems and beyond,
Lyon, 24 October 2017

Outline

- Linear quadlattice & multidimensional consistency;
- Lagrangian 2-form structure;
- The quantum lattice 2-form structure (quadratic case);
- Periodic Reduction;
- Lagrangian 1-form structure;
- Quantum propagators & quantum 1-form structure.

Key references:

1 S.B. Lobb and F.W. Nijhoff, *Langrangian multiforms and multidimensional consistency*, J. Phys. A: Math & Theor **42** (2009) 454013.

2 S.B. Lobb and F.W. Nijhoff, *A variational principles for discrete integrable systems*, arXiv: 1312.1440v2, (submitted to SIGMA).

3 S. King and F.W. Nijhoff, *Quantum variational principle and quantum multiform structure: the case of quadratic Lagrangians*, arXiv: 1702.08709, (submitted to Annals of Physics).

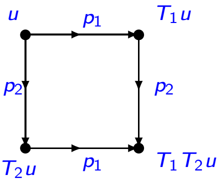
Multidimensional consistency (MDC) on the lattice

Quadrilateral PΔEs on the 2D lattice:

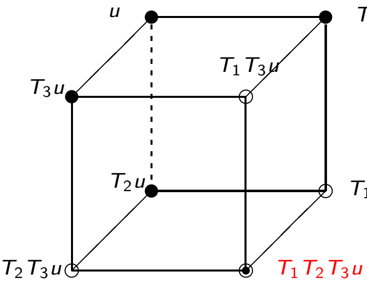
$$\mathbf{Q}(u, T_1 u, T_2 u, T_1 T_2 u; p_1, p_2) = 0$$

notation of shifts on the elementary quadrilateral on a rectangular lattice:

$$u := u(n_1, n_2), \;\; T_1 u = u(n_1+1, n_2)$$
$$T_2 u := u(n_1, n_2+1), \;\; T_1 T_2 u = u(n_1+1, n_2+1)$$



Consistency-around-the cube:



Verifying consistency: Values at the black disks are initial values, values at open circles are uniquely determined from them, but there are three different ways to compute $T_1 T_2 T_3 u$.

Examples

Using the abbreviations:

$$\widetilde{u} = T_1 u, \quad \widehat{u} = T_2 u, \quad \widehat{\widetilde{u}} = T_1 T_2 u,$$

associated with parameters $p_1 =: p$, $p_2 =: q$ respectively, we have

Linear quad-equation

$$\mathbf{Q}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = (p-q)(\widehat{u} - \widetilde{u}) - (p+q)(\widehat{\widetilde{u}} - u) \ ,$$

lattice potential KdV eq. (H1)

$$\mathbf{Q}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; p, q) = (p - q + \widehat{u} - \widetilde{u})(p + q + u - \widehat{\widetilde{u}}) - p^2 + q^2 \ ,$$

lattice Krichever-Novikov eq. (Q4) [Adler, 1998]

$$\begin{aligned} \mathbf{Q}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}; \alpha, \beta) &= p(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - q(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) \\ &\quad - r(u\widehat{\widetilde{u}} + \widehat{u}\widetilde{u}) + pqr(1 + u\widetilde{u}\widehat{u}\widehat{\widetilde{u}}) \ , \end{aligned}$$

where the lattice parameters are in terms of Jacobi elliptic functions:
$p = \sqrt{k}\,\mathrm{sn}(\alpha; k)$, $q = \sqrt{k}\,\mathrm{sn}(\beta; k)$, $r = \sqrt{k}\,\mathrm{sn}(\alpha - \beta; k)$.

Consistency-around-the-cube is by direct computation for these examples. In the linear case it boils down to the key partial fraction identity

$$S_{p,q}S_{r,q} + S_{p,r}S_{q,r} + S_{p,r}S_{p,q} = 1 \ , S_{p,q} := \frac{p+q}{p-q} \ .$$

MDC implies many main properties of the quad-equation (Lax pair, Bäcklund transforms, exact (e.g. soliton type) solutions).

Main question: How to capture MDC in terms of a variational principle?

This question was answered in a series of recent papers:

• S. Lobb & FWN: *Lagrangian multiforms and multidimensional consistency*, J. Phys. A:Math Theor. **42** (2009) 454013

• S. Lobb, FWN & R. Quispel, *Lagrangian multiform structure for the lattice KP system*, J. Phys. A:Math Theor. **42** (2009) 472002

• S. Lobb & FWN, *Lagrangian multiform structure for the lattice Gel'fand-Dikii hierarchy*, J. Phys. A:Math. Theor. **43** (2010) 072003

• A. Bobenko and Yu. Suris, *On the Lagrangian Structure of Integrable Quad-Equations*, Lett. Math. Phys. **92** (2010) 17–31

• P. Xenitidis, FWN & S. Lobb, *On the Lagrangian formulation of multidimensionally consistent systems*, Proc. Roy. Soc. **A467** # 2135 (2011) 3295-3317

• S. Yoo-Kong, S. Lobb and FWN, *Discrete-time Calogero-Moser system and Lagrangian 1-form structure*, J. Phys. A: **44** (2011) 365203

• J. Atkinson, S.B. Lobb and FWN, *An integrable multicomponent quad equation and its Lagrangian formalism*, Theor. Math. Phys 173 (2012) # 3 pp. 1644-1653

• S. Yoo-Kong and FWN, *Discrete-time Ruijsenaars-Schneider system and Lagrangian 1-form structure*, arXiv:1112.4576 (Dec. 2011)

• Yu. Suris, *Variational formulation of commuting Hamiltonian flows: multi-time Lagrangian 1-forms*, J. of Geometric Mechanics **5** (2013)

• R. Boll, M. Petrera and Yu. Suris, *What is integrability of discrete variational systems?* Proc. Roy. Soc. **470** (2013).

• A.Bobenko and Yu. Suris, *Discrete pluriharmonic functions as solutions of linear pluri-Lagrangian systems*, CMP **336** (2015) 199.

Lagrangian 2-form structure (classical case)

The quad-equation can be consistently embedded into a *multidimensional* lattice, with directions labelled by subscripts $i, j, k, \ldots$:

$$(p_i + p_j)(T_i u - T_j u) = (p_i - p_j)(u - T_i T_j u)$$

This has the consistency-around-the-cube property.
This and other quad equations come from a variational principle on the 3-point Lagrangian:

$$\mathscr{L}_{ij}(u, T_i u, T_j u) := \mathscr{L}(u, T_i u, T_j u; p_i, p_j) = u(T_i u - T_j u) - \frac{1}{2}\frac{p_i + p_j}{p_i - p_j}(T_i u - T_j u)^2$$

The discrete Euler-Lagrange (EL) equation

$$\frac{\partial}{\partial u}\left(\mathscr{L}_{i,j}(u, T_i u, T_j u) + \mathscr{L}_{i,j}(T_i^{-1} u, u, T_j T_i^{-1} u) + \mathscr{L}_{i,j}(T_j^{-1} u, T_j^{-1} T_i u, u)\right) = 0$$

gives a weak form of the quad-equation: sum of two copies shifted over a diagonal.
Key observation: The Lagrangian above possesses a *closure property*:

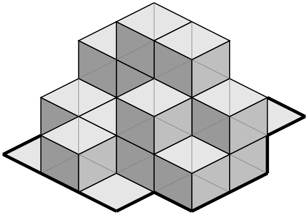
$$\Delta_1 \mathscr{L}_{23}(u) + \Delta_2 \mathscr{L}_{31}(u) + \Delta_3 \mathscr{L}_{12}(u) = 0$$

on the solutions of the quad-equation. Here $\Delta_i = T_i - \mathrm{id}$ is the difference operator in the i^{th} direction.

• The closure property (which holds for all MDC quad equations) holds the key to a new variational description which allows one to capture multidimensional consistency in a Lagrangian framework using **Lagrangian multi-forms**.

Classical Two-form Structure

We define the action as the sum of the Lagrangians over a surface σ



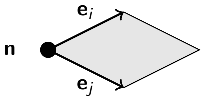
Action functional on a discrete (oriented) surface :

$$S[u(\mathbf{n});\sigma] = \sum_{\sigma_{ij}(\mathbf{n})\in\sigma} \mathscr{L}_{ij}(\mathbf{n})$$

where $\mathscr{L}_{ij}(\mathbf{n}) = -\mathscr{L}_{ji}(\mathbf{n})$ has the interpretation of a discrete Lagrangian 2-form:
These are antisymmetric expressions of the form:

$$\mathscr{L}_{ij}(u(\mathbf{n})) = \mathscr{L}(u(\mathbf{n}), u(\mathbf{n}+\mathbf{e}_i), u(\mathbf{n}+\mathbf{e}_j); p_i, p_j)$$

defined on elementary oriented plaquettes, in a multidimensional lattice, characterized by ordered triplets $\sigma_{ij}(\mathbf{n}) = (\mathbf{n}, \mathbf{n}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}_j)$



In addition to variations of the dependent variables $u \to u + \delta u$, we also require the action *to be stationary under variations of the surface.*

Lattice action for the closed cube surface

To derive elementary configurations we need action over the (decorated) full oriented cube:

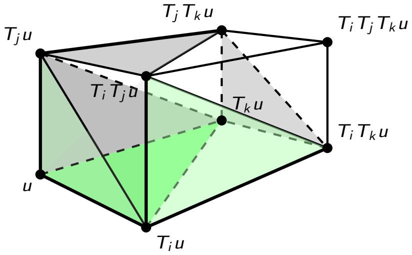


Figure : Decorated cube.

This gives rise to a lattice action functional:

$$S[u; cube] = L_{i,j}(u, T_iu, T_ju) + L_{j,k}(u, T_ju, T_ku) + L_{k,i}(u, T_ku, T_iu)$$
$$-L_{i,j}(T_ku, T_iT_ku, T_jT_ku) - L_{j,k}(T_iu, T_iT_ju, T_iT_ku) - L_{k,i}(T_ju, T_jT_ku, T_iT_ju).$$

The faces joining each vertex involved in the action will give rise to the various elementary surface configurations: the elementary actions that will lead to the fundamental system of EL equations.

Elementary configurations for lattice action

Over curved quad-surfaces we need the following types of elementary configurations:

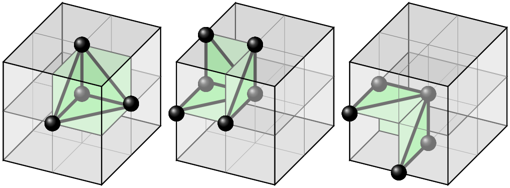


Figure : Elementary lattice configurations in 3D.

The action functionals corresponding to these configurations give rise to the fundamental system of EL equations *for the Lagrangian*:

$$(EL1) \qquad \frac{\partial}{\partial u}\Big(\mathscr{L}_{i,j}(u, T_i u, T_j u) + \mathscr{L}_{j,k}(u, T_j u, T_k u) + \mathscr{L}_{k,i}(u, T_k u, T_i u)\Big) = 0,$$

$$(EL2) \qquad \frac{\partial}{\partial u}\Big(\mathscr{L}_{i,j}(T_i^{-1}u, u, T_i^{-1}T_j u) - \mathscr{L}_{j,k}(u, T_j u, T_k u) + \mathscr{L}_{k,i}(T_i^{-1}u, T_i^{-1}T_k u, u)\Big) = 0,$$

$$(EL3) \qquad \frac{\partial}{\partial u}\Big(\mathscr{L}_{j,k}(T_j^{-1}(u), u, T_j^{-1}T_k u) + \mathscr{L}_{k,i}(T_i^{-1}u, T_i^{-1}T_k u, u)\Big) = 0.$$

(up to permutations of the lattice indices).

Furthermore, imposing that the action remains invariant under (discrete) deformations of the surface (allowing the above equations to hold simultaneously) the system is supplemented with the closure relation:

$$(EL4) \quad \Delta_i\mathscr{L}(u, T_j u, T_k u; p_j, p_k) + \Delta_j\mathscr{L}(u, T_k u, T_i u; p_k, p_j) + \Delta_k\mathscr{L}(u, T_i u, T_j u; p_i, p_j) = 0\ .$$

Main hypothesis: *The solutions of the above linear system of equations for the Lagrangians* $\mathscr{L}_{i,j}$ *correspond exactly to the Lagrangian components for* **integrable** *(in the sense of multidimensional consistency) quad-lattice systems.*

Possible New Foundational principle: *Lagrangians of a fundamental theory should emerge as solutions of a least-action principle involving not only criticality w.r.t. variations in the dependent variables, but also w.r.t. the variations of the geometry in the independent variables. I.o.w. they should arise as solutions of set of extended EL equations, i.e. emerge from the variational principle itself, rather than being posed by tertiary considerations.*

Analysis of the EL system

Analysing the fundamental EL system (EL1)-(EL4) under the assumption that $u,\, T_i u,\, T_j u,\, T_k u$ are independent and can be chosen arbitrarily, and considering the functional dependence on the arbitrary parameters p_i, p_j, p_k we arrive at the following:

Theorem

Suppose $u, T_i u, T_j u, T_k u$ are independent and can be chosen arbitrarily. Eq (EL1) implies that the anti-symmetric Lagrangian $\mathscr{L}_{i,j} = (u, T_i u, T_j u; p_i, p_j)$ has the form

$$\mathscr{L}_{i,j}(u, T_i u, T_j u) = A_i(u, T_i u) - A_j(u, T_j u) + B_{i,j}(T_i u, T_j u),$$

where $A_i(u, T_i u) = A(u, T_i u; p_i)$, and $B_{i,j}(T_i u, T_j u) = B(T_i u, T_j u; p_i, p_j)$ for some functions A, B of the arguments and lattice parameters, where $B_{i,j} = -B_{j,i}$ is antisymmetric in the i, j-arguments.

Furthermore, from the eqs (EL2), (EL3) one can deduce the following:

Theorem

The Euler-Lagrange equations (EL2),(EL3) determine the following relation on each single quad:

$$(QEL) \qquad \frac{\partial}{\partial u}\left(\mathscr{L}_{i,j}(T_i^{-1}u, u, T_i^{-1}T_j u)\right) = \frac{\partial}{\partial u}\left(A_j(u, T_j u)\right)\,,$$

where A_j is determined up to an (direction-independent) function $h(u)$, which w.l.o.g. can be absorbed into A_j.

The latter equation, in fact, leads directly to the quadrilateral lattice equation $Q(u, T_i u, T_j u, T_i T_j u; p_i, p_j) = 0$.

Example: quadratic 3-point Lagrangian 2-forms

Let us consider the general homogeneous quadratic Lagrangian 2-form component, which must be of the form:

$$\mathscr{L}_{i,j}(u, T_i u, T_j u) = A_i(u, T_i u) - A_j(u, T_j u) + B_{i,j}(T_i u, T_j u) \ ,$$

by setting:

$$A_i(u, T_i u) = \tfrac{1}{2} a_i u^2 + a_i' u T_i u + \tfrac{1}{2} a_i'' (T_i u)^2 \ , \quad B_{i,j} = \tfrac{1}{2} b_{ij} (T_i u)^2 - \tfrac{1}{2} b_{ji} (T_j u)^2 + b_{ij}' (T_i u) T_j u \ ,$$

where $b_{ji}' = -b_{ij}'$. Applying the eq. (QEL) in the form:

$$\frac{\partial}{\partial T_i u}\Big(\mathscr{L}_{ij}(u, T_i u, T_j u) \Big) = \frac{\partial}{\partial T_i u}\Big(A_j(T_i u, T_i T_j u) \Big) \ ,$$

(which holds for all directions i, j) we obtain the linear quad-equation:

$$a_i' u + (a_i'' + b_{ij} - a_j) T_i u + b_{ij}' T_j u = a_j' T_i T_j u \ .$$

Since these hold for arbitrary i, j-labels we obtain the conditions:

$$a_i'^2 = a_j'^2 \ , \ \text{and} \ \ (a_i'' + b_{ij} - a_j) a_i' = a_j' b_{ji}' \ .$$

Setting $a_i' = a_j' =: a'$ and implementing the other condition we get the quad equation: $T_i T_j u = u + \frac{1}{a'} b_{ij}' (T_j u - T_i u)$, with Lagrangian:

$$\mathscr{L}_{i,j} = \tfrac{1}{2}(a_i - a_j) u^2 + a' u (T_i u - T_j u) + \tfrac{1}{2} a_j (T_i u)^2 - \tfrac{1}{2} a_i (T_j u)^2 - \tfrac{1}{2} b_{ij}' (T_i u - T_j u)^2 \ ,$$

where the terms with a_i can be removed w.l.o.g., and we can set $a' = 1$. The closure relation (EL4) leads to the functional relation

$$b_{ij}'(b_{ik}' - b_{jk}') = 1 - b_{ik}' b_{jk}' \quad \Rightarrow \quad b_{ij}' = (1 - P_i P_j)/(P_i - P_j) \ ,$$

in terms of new lattice parameter $P_i := b_{i,k_0}'$ (with k_0 fixed).

Universal Quadrilateral 3-point Lagrangian

Under the assumption that resulting quad equation is an *affine-linear* equations for a scalar dependent variable $u = u(\mathbf{n})$ of the quad-lattice:

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u, u_i, u_j, u_{ij}) = 0 \quad , \quad u_i := u(\mathbf{n} + \mathbf{e}_i) \ , \quad u_{ij} := u(\mathbf{n} + \mathbf{e}_i + \mathbf{e}_j) \ ,$$

(where $\mathfrak{p}_i$, $\mathfrak{p}_j$, and where the quad function Q possesses the symmetries of the square:

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u, u_i, u_j, u_{ij}) = Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u_i, u, u_{ij}, u_j) = Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u_j, u_{ij}, u, u_i) = -Q_{\mathfrak{p}_j,\mathfrak{p}_i}(u, u_j, u_i, u_{ij})$$

the general solution (up to constant direction-independent factor) for the Lagrangians was found [P. Xenitidis, F.W. Nijhoff & S. Lobb,Proc. Roy. Soc. **467** (2011)]:

$$\begin{aligned}\mathscr{L}(u, u_i, u_j; \mathfrak{p}_i, \mathfrak{p}_j) \quad = \quad & \int_{u^0}^{u}\int_{u_i^0}^{u_i} \frac{dx\,dy}{h_{\mathfrak{p}_i}(x,y)} - \int_{u^0}^{u}\int_{u_j^0}^{u_j} \frac{dx\,dy}{h_{\mathfrak{p}_j}(x,y)} - \int_{u_i^0}^{u_i}\int_{u_j^0}^{u_j} \frac{dx\,dy}{h_{\mathfrak{p}_{ij}}(x,y)} \\ & + \int_{u_i^0}^{u_i} dx \int_{u_j^0}^{Y(u^0,x,u_{ij}^0)} \frac{dy}{h_{\mathfrak{p}_{ij}}(x,y)} + \int_{u_j^0}^{u_j} dy \int_{u_i^0}^{X(u^0,y,u_{ij}^0)} \frac{dx}{h_{\mathfrak{p}_{ij}}(x,y)}\end{aligned}$$

where the limiting functions X and Y are solutions of the equations

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u^0, x, Y, u_{ij}^0) = 0 \quad \text{respectively} \quad Q_{\mathfrak{p}_i,\mathfrak{p}_j}(u^0, X, y, u_{ij}^0) = 0 \ .$$

Here the denominators $h_{\mathfrak{p}}(x,y)$ in the integrand are biquadratic functions associated with Q, defined by the discriminant relations:

$$Q_{u_j} Q_{u_{ij}} - Q\, Q_{u_j u_{ij}} =: K_{\mathfrak{p}_i,\mathfrak{p}_j} h_{\mathfrak{p}_i}(u, u_i) \ , \quad Q_{u_i} Q_{u_{ij}} - Q\, Q_{u_i u_{ij}} =: K_{\mathfrak{p}_j,\mathfrak{p}_i} h_{\mathfrak{p}_j}(u, u_j)$$
$$\text{and} \quad Q_u Q_{u_{ij}} - Q\, Q_{u u_{ij}} =: -K_{\mathfrak{p}_i,\mathfrak{p}_j} h_{\mathfrak{p}_{ij}}(u_i, u_j)$$

where $K_{\mathfrak{p},\mathfrak{q}} = -K_{\mathfrak{q},\mathfrak{p}}$ is a function of the lattice parameters $\mathfrak{p}$, $\mathfrak{q}$ alone ($\mathfrak{p}_{ij}$ is related to $\mathfrak{p}_i$, $\mathfrak{p}_j$ through an addition formula on an algebraic curve).
Remark: The Lagrangian $\mathscr{L}$ is in fact a *interpolating Lagrangian* in that the EL equations yield the relevant quad equation in both u and u^0!

Q4 (lattice Krichever-Novikov) equation

This equation, due to V.Adler (1998) reads:

$$Q_{\mathfrak{p}_i,\mathfrak{p}_j} = p_i(u\,u_i + u_j u_{ij}) - p_j(u\,u_j + u_i u_{ij}) \tag{1}$$

$$-p_{ij}(u\,u_{ij} + u_i u_j) + p_i p_j p_{ij}(1 + u\,u_i u_j u_{ij}) \tag{2}$$

where $p_i = \sqrt{k}\,\mathrm{sn}(\alpha_i;k)$, $p_j = \sqrt{k}\,\mathrm{sn}(\alpha_j;k)$, $p_{ij} = \sqrt{k}\,\mathrm{sn}(\alpha_{ij};k)$ with $\alpha_{ij} = \alpha_i - \alpha_j$. I.o.w. the lattice parameters $\mathfrak{p} = (p,P) = (\mathrm{sn}(\alpha;k), \sqrt{k}\mathrm{sn}'(\alpha;k))$ are points on an elliptic curve in this case. Q4 has arisen as the top equation (master equation) in the classification of scalar affine-linear quad equations [V.Adler, A.Bobenko & Yu. Suris, CMP (2003)]
For the biquadratics we have

$$h_{\mathfrak{p}}(x,y) = p(1 + x^2y^2) - \frac{1}{p}(x^2 + y^2) + 2\frac{P}{p}xy \quad , \quad K_{\mathfrak{p}_i,\mathfrak{p}_j} = -p_i p_j p_{ij}$$

where $\mathfrak{p} = (p,P)$ are points on the elliptic curve given by $P^2 = p^4 - (k + 1/k)p^2 + 1$ and k the modulus of the Jacobi elliptic function. The double integral in the Lagrangian can be evaluated as:

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{dx\,dy}{h_{\mathfrak{p}}(x,y)} = -2\int_{\eta_0}^{\eta_1} d\eta\, \log\left(\frac{\mathrm{sn}(\xi_1) - \mathrm{sn}(\eta+\alpha)}{\mathrm{sn}(\xi_1) - \mathrm{sn}(\eta-\alpha)}\,\frac{\mathrm{sn}(\xi_0) - \mathrm{sn}(\eta-\alpha)}{\mathrm{sn}(\xi_0) - \mathrm{sn}(\eta+\alpha)}\right) ,$$

with $x_i = \sqrt{k}\,\mathrm{sn}(\xi_i;k)$, $y_i = \sqrt{k}\,\mathrm{sn}(\eta_i;k)$ and $p = \sqrt{k}\,\mathrm{sn}(\alpha;k)$.
the latter is an elliptic variant of the dilogarithm function.

Variational equations for continuous Lagrangian 2-forms

Choosing a parametrisation of the surface

$$\sigma : \quad \mathbf{p} = \mathbf{p}(s,t) = (p_i(s,t)) \ , \quad (s,t) \in \Omega \subset \mathbb{R}^2 \ ,$$

where Ω is some open domain in the space of parameters s, t, we can write for the action:

$$S[u(\mathbf{p});\sigma] = \int_\sigma \sum_{i<j} \mathscr{L}_{i,j} \mathrm{d}p_i \wedge \mathrm{d}p_j = \iint_\Omega \sum_{i<j} \left(\mathscr{L}_{i,j} \frac{\partial(p_i,p_j)}{\partial(s,t)} \right) \mathrm{d}s\, \mathrm{d}t \ ,$$

We have two types of variations:

• Variations of the surface: $\sigma \to \sigma + \sigma$, (i.e., making a infinitesimal variations $\mathbf{p} \mapsto \mathbf{p} + \delta\mathbf{p}$, in the parametrisation). The closure relation can be obtained by considering the Lagrangian as a function of the independent variables

$$\mathsf{L}(\mathbf{p}(s,t)) := \sum_{i<j} \left(\mathscr{L}_{i,j} \frac{\partial(p_i,p_j)}{\partial(s,t)} \right) \ ,$$

and apply the usual EL equations:

$$\frac{\delta \mathsf{L}}{\delta \mathbf{p}(s,t)} = 0 \quad \Rightarrow \quad \partial_{p_i}\mathscr{L}_{j,k} + \partial_{p_j}\mathscr{L}_{k,i} + \partial_{p_k}\mathscr{L}_{i,j} = 0 \ .$$

• Infinitesimal variations of the dependent variable $u \mapsto u + \delta u$, on an arbitrary, but fixed, surface. This has two contributions:

♢ tangential contributions, i.e. from components $(\nabla \delta u)_{\|}$ along the surface;

♢ orthogonal contributions, i.e. from components $(\nabla \delta u)_{\perp}$ orthogonal to the surface.

Lagrange 2-form in 3D space

In the simple case of smooth 2D surfaces σ embedded in $\mathbb{R}^3$, and $\mathscr{L}$ depending only on the first jet, we get the following set of equations [1]:

- From the tangential contributions:

$$\sum_{i<j}\left[\frac{\partial(p_i,p_j)}{\partial(s,t)}\frac{\partial\mathscr{L}_{i,j}}{\partial u}-\frac{\partial}{\partial s}\left(\frac{\partial(p_i,p_j)}{\partial(s,t)}\frac{\mathbf{p}_t\times\mathbf{n}}{\|\mathbf{p}_s\times\mathbf{p}_t\|}\cdot\frac{\partial\mathscr{L}_{i,j}}{\partial\nabla u}\right)\right.$$
$$\left.+\frac{\partial}{\partial t}\left(\frac{\partial(p_i,p_j)}{\partial(s,t)}\frac{\mathbf{p}_s\times\mathbf{n}}{\|\mathbf{p}_s\times\mathbf{p}_t\|}\cdot\frac{\partial\mathscr{L}_{i,j}}{\partial\nabla u}\right)\right]=0$$

where $\mathbf{n}$ is the unit normal to the surface, and:

- From the transversal contributions:

$$\sum_{i<j}\frac{\partial(p_i,p_j)}{\partial(s,t)}\,\mathbf{n}\cdot\frac{\partial\mathscr{L}_{i,j}}{\partial\nabla u}=0\ .$$

Example: Scalar field Lagrangian giving rise to MDC system of higher-order equations [FWN, A Hone, N Joshi, 2001] :

$$\mathscr{L}_{ij}=\tfrac{1}{4}(p_i^2-p_j^2)\frac{(\partial_{p_i}\partial_{p_j}u)^2}{(\partial_{p_i}u)\partial_{p_j}u}+\frac{1}{p_i^2-p_j^2}\left(n_i^2p_i^2\frac{\partial_{p_j}u}{\partial_{p_i}u}+n_j^2p_j^2\frac{\partial_{p_i}u}{\partial_{p_j}u}\right)\ ,$$

in terms of only u and its derivatives. The Euler-Lagrange equation:

$$\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\left(\frac{\partial\mathscr{L}_{ij}}{\partial(\partial_{p_i}\partial_{p_j}u)}\right)-\frac{\partial}{\partial p_i}\left(\frac{\partial\mathscr{L}_{ij}}{\partial(\partial_{p_i}u)}\right)-\frac{\partial}{\partial p_j}\left(\frac{\partial\mathscr{L}_{ij}}{\partial(\partial_{p_j}u)}\right)=0\ ,$$

yields a generalization of the Ernst equations of General Relativity, cf. [A Tongas, D Tsoubelis, P Xenitidis, 2002]

Quantisation of the Lattice Equation

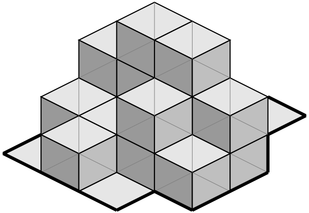
Goal: to apply a multiform path integral approach to quantising the linear lattice equation.

$$(p_i + p_j)(u_i - u_j) = (p_i - p_j)(u - u_{ij})$$

Lagrangian:

$$\mathscr{L}_{ij}(u, u_i, u_j; p_i, p_j) = u(u_i - u_j) - \frac{1}{2} s_{ij}(u_i - u_j)^2 \ ; \qquad s_{ij} = \frac{p_i + p_j}{p_i - p_j}$$

Quantum field theory: discretised space-time, Lagrangian in two dimensions over field variables $u(\mathbf{n})$ indexed by discrete vector $\mathbf{n}$.



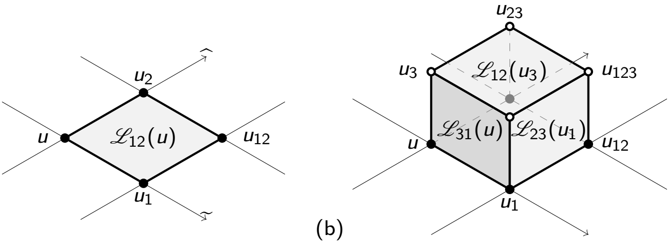
Space-time boundary $\partial\sigma$ enclosing surface σ.
Action: $\mathscr{S}[u_{n,m}; \sigma] = \sum_\sigma \mathscr{L}(\mathbf{n})$.
Propagator (all interior field variables are integrated over):

$$K_\sigma(\partial\sigma) = \int [\mathscr{D}u_{n,m}] \ e^{i\mathscr{S}[u_{n,m};\sigma]/\hbar}$$
$$= \mathscr{N}_\sigma \prod_{\mathbf{n}\in\sigma} \int \mathrm{d}u(\mathbf{n}) \ e^{i\mathscr{S}[u(\mathbf{n});\sigma]/\hbar}$$

Normalisation and regularisation, denoted by $\mathscr{N}_\sigma$.

Surface-independence: the pop-up cube

Main question: What happens to propagator $K_\sigma(\partial\sigma)$ under variation of the surface σ?



$$\mathscr{S}_{pop}[u_{n,m}] = \mathscr{L}_{23}(u_1) + \mathscr{L}_{31}(u_2) + \mathscr{L}_{12}(u_3) - \mathscr{L}_{23}(u) - \mathscr{L}_{31}(u)$$

$$K_{pop} = \mathscr{N}_\sigma \iiiint \mathrm{d}u_3\mathrm{d}u_{31}\mathrm{d}u_{23}\mathrm{d}u_{123}\ \exp\left(\frac{i}{\hbar}\mathscr{S}_{pop}[u_{n,m}]\right)$$

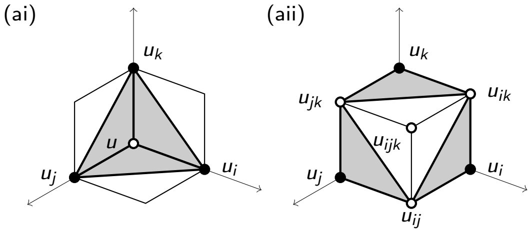
$$K_\sigma = \mathscr{N}_\sigma \exp\left(\frac{i}{\hbar}\mathscr{L}_{12}(u,u_1,u_2)\right) \qquad = \mathscr{N}_\sigma V^2 \frac{2\pi\hbar}{s_{23}} \exp\left(\frac{i}{\hbar}\mathscr{L}_{12}(u,u_1,u_2)\right)$$

Performing in the quadratic case the computations of Gaussian integrals, we find: *the contributions to the propagator from each surface are (up to normalising factor) the same*. Thus, *the propagator invariant under this surface-move!*

Elementary surface moves

The pop-up cube suggests we have surface-independence in all surface moves. Like in the classical case we consider three *elementary moves* that form the basis of the possible surface deformations.

Move (a):



Comparing (ai):

$$\mathscr{S}_{(ai)} = \mathscr{L}_{ij}(u) + \mathscr{L}_{jk}(u) + \mathscr{L}_{ki}(u) \ ; \quad K_{(ai)} = \int \mathrm{d}u \exp\left[i\mathscr{S}_{(ai)}/\hbar\right]$$

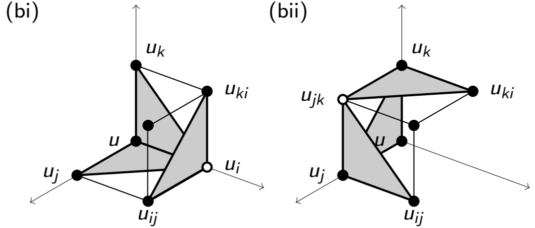
with (aii):

$$\mathscr{S}_{(aii)} = \mathscr{L}_{ij}(u_k) + \mathscr{L}_{jk}(u_i) + \mathscr{L}_{ki}(u_j) \ ; \quad K_{(aii)} = \iiiint \mathrm{d}u_{ij}\mathrm{d}u_{jk}\mathrm{d}u_{ki}\mathrm{d}u_{ijk} \exp\left[i\mathscr{S}_{(aii)}/\hbar\right]$$

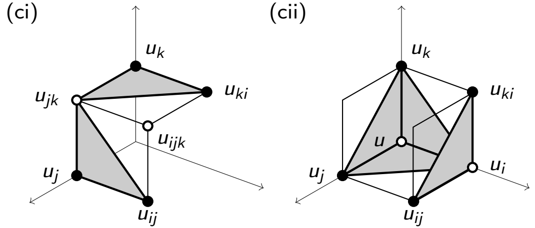
it follows the resulting propagators in $K_{(ai)}$ and $K_{(aii)}$ are the same.

Similar results for the other elementary moves.

Move (b):



Move (c):



Similar calculations, performing Gaussian integrals over the intermediate points, can be done for moves (b) and (c), leading to identities between corresponding propagators.
Conclusion: For our choice of Lagrangian, the propagator $K_\sigma(\partial\sigma)$ is *independent* of the surface: it depends only on the boundary.

Quantum variational principle for surfaces

• Propagator for general quadratic Lagrangian 2-form over discrete surface σ, with action $\mathscr{S}[u(\mathbf{n});\sigma]$ as defined before;

$$K_\sigma(\partial\sigma) = \int [\mathscr{D}u_{n,m}]\ e^{i\mathscr{S}[u_{n,m};\sigma]/\hbar} \ := \mathscr{N}_\sigma \prod_{\mathbf{n}\in\sigma} \int \mathrm{d}u(\mathbf{n})\ e^{i\mathscr{S}[u(\mathbf{n});\sigma]/\hbar}\ .$$

In general $K_\sigma(\partial\sigma)$ is a function of the field variables on the boundary $\partial\sigma$ and also depends on the surface σ itself;

• For a special choice of discrete Lagrangian 2-form the propagator $K_\sigma(\partial\sigma)$ is *independent of the surface* σ. This Lagrangian exists at a *critical point* of the variation of the surface, such that some of the integrations over field variables reduce to volume factors;

• The condition of stationarity of propagator under surface moves determines (up to equivalence) the Lagrangian form (this has been demonstrated for the case of quadratic 3-point Lagrangians), leading to Lagrangian of the form:

$$\mathscr{L}(u,u_i,u_j;p_i,p_j) = u(u_i-u_j) - \frac{1}{2}s_{ij}(u_i-u_j)^2\ ; \qquad s_{ij} = \frac{p_i+p_j}{p_i-p_j}\ .$$

• The invariance under surface deformation suggests that one could consider a novel quantum object obtained by a *sum over all surfaces*,

$$K(\partial\sigma) = \sum_{\sigma\in\mathfrak{S}} \mathscr{N}_\sigma K_\sigma(\partial\sigma)$$

leading to a functional of the Lagrangian 2-form components, and which attains a critical point for Lagrangians for which the usual surface-dependent propagator $K_\sigma(\partial\sigma)$ becomes invariant.

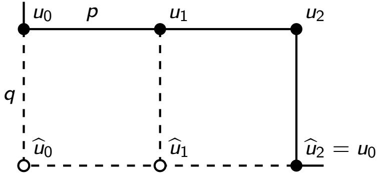
One Dimensional Reduction: Discrete Harmonic Oscillator

Reduce quad equations to a *finite-dimensional dynamical map* imposing (periodic) initial value problem.

Simplest non-trivial example: reduction of the linear quad-equation

$$(p+q)(\widehat{u}-\widetilde{u}) = (p-q)(\widehat{\widetilde{u}}-u)\ ,$$

on the periodically repeated initial value configuration:



Imposing initial data u_0, u_1 and u_2, and letting $\widehat{u}_2 = u_0$, we obtain from the quad-equation the dynamical map: $(u_0, u_1, u_2) \to (\widehat{u}_0, \widehat{u}_1, \widehat{u}_2)$:

$$\widehat{u}_0 = u_1 + s(\widehat{u}_1 - \widehat{u}_2)\ , \quad \widehat{u}_1 = u_2 + s(u_0 - u_1)\ , \quad \widehat{u}_2 = u_0\ ; \quad s := \frac{p-q}{p+q}$$

This is a finite-dimensional discrete system; introducing the reduced variables:

$$x := u_1 - u_0\ , \qquad y := u_2 - u_1$$

and, by eliminating y, write the second order ordinary difference equation (OΔE):

$$\widehat{x} + 2bx + \underset{\frown}{x} = 0\ , \quad b := 1 + 2s - s^2\ ,$$

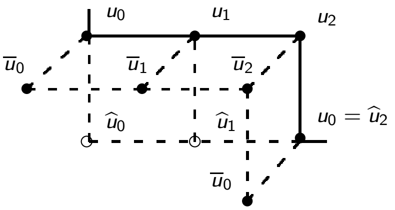
where $\underset{\frown}{x}$ denotes the reverse shift to $\widehat{x}$, is a discrete harmonic oscillator.

Commuting discrete flow

To construct a commuting discrete flow we consider the dynamics in an additional direction of the lattice, described by lattice equation:

$$(p+r)(\overline{u}-\widetilde{u}) = (p-r)(\widetilde{\overline{u}}-u)\ ,\quad (q+r)(\overline{u}-\widehat{u}) = (q-r)(\widehat{\overline{u}}-u)\ .$$

Consider the diagram:



Map in the additional direction:

$$\begin{aligned} \overline{u}_0 &= u_1 + t(\overline{u}_1 - u_0)\ , \\ \overline{u}_1 &= u_2 + t(\overline{u}_2 - u_1)\ , \\ \overline{u}_2 &= u_0 + t'(\overline{u}_0 - u_2)\ . \end{aligned}$$

Here:

$$t := \frac{p-r}{p+r}\ ,\quad t' := \frac{q-r}{q+r}\ .$$

In terms of the reduced variables $x = u_1 - u_0$, $y = u_2 - u_1$, we get the map $(x, y) \to (\overline{x}, \overline{y})$ given by:

$$\overline{x} = y + t(\overline{y} - x)\ ,\quad \overline{y} = -y - \frac{1-t}{1-tt'}(x + t'\overline{x})$$

leading to

$$\overline{x} + 2ax + \underline{x} = 0\ ,\quad \text{with}\quad 2a := \frac{(2t+1-t^2) - t'(2t-1+t^2)}{1-t^2t'}$$

The maps $(x, y) \to (\widehat{x}, \widehat{y})$ and $(x, y) \to (\overline{x}, \overline{y})$ commute provided we have the key identity on the parameters: $stt' = s - t + t'$.

Corner equations & Lagrangians

Our parametrisation is slightly simplified by introducing the parameters $P := p^2 + pq$, $Q := q^2$ and $R := r^2$, in terms of which

$$a = \frac{P-R}{P+R} \;, \quad b = \frac{P-Q}{P+Q}$$

By combining the maps $(x,y) \to (\widehat{x},\widehat{y})$ and $(x,y) \to (\overline{x},\overline{y})$ and eliminating y in a different way, we can derive *corner equations* for the evolution, linking x, $\widehat{x}$ and $\overline{x}$; or $\widehat{x}$, $\overline{x}$ and $\widehat{\overline{x}}$ respectively. Thus:

$$\begin{aligned} \left(\frac{P-Q}{q} - \frac{P-R}{r}\right) x &= \frac{P+R}{r}\overline{x} - \frac{P+Q}{q}\widehat{x} \\ \left(\frac{P-Q}{q} - \frac{P-R}{r}\right) \widehat{\overline{x}} &= \frac{P+R}{r}\widehat{x} - \frac{P+Q}{q}\overline{x} \end{aligned}$$

Lagrangian 1-form structure: On the level of the Lagrangian description the commutativity of the flows $x \to \widehat{x}$ and $x \to \overline{x}$ is described by the *Lagrangian 1-form structure*. Here, the relevant Lagrangians

$$\begin{aligned} \mathscr{L}_a(x,\overline{x}) &= \frac{P-R}{r}\left[x^2 + \overline{x}^2 + \frac{2}{a}x\overline{x}\right] \;, \quad a = \frac{P-R}{P+R} \;, \\ \mathscr{L}_b(x,\widehat{x}) &= \frac{P-Q}{q}\left[x^2 + \widehat{x}^2 + \frac{2}{b}x\widehat{x}\right] \;, \quad b = \frac{P-Q}{P+Q} \;, \end{aligned}$$

should be regarded as *components of a difference 1-form* each associated with a a direction on a 2D lattice (with given orientation). It obeys, on solutions of the equations of motion the closure relation:

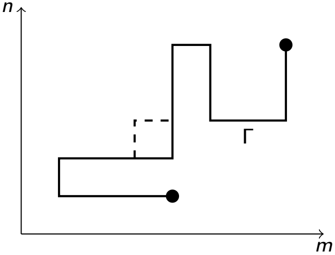
$$\square\mathscr{L} := \mathscr{L}_a(\widehat{x},\widehat{\overline{x}}) - \mathscr{L}_a(x,\overline{x}) - \mathscr{L}_b(\overline{x},\widehat{\overline{x}}) + \mathscr{L}_b(x,\widehat{x}) = 0$$

1-form Action functional

The action functional is defined as the sum [Yoo-Kong, Lobb, FWN, 2010]:

$$\mathscr{S}[x(\mathbf{n});\Gamma] = \sum_{\gamma(\mathbf{n})\in\Gamma} \mathscr{L}_i(x(\mathbf{n}), x(\mathbf{n}+\mathbf{e}_i)) \ ,$$

over an arbitrary discrete curve Γ in the space of independent discrete variables m, n, consisting of connected oriented links. $\gamma_i(\mathbf{n}) = (\mathbf{n}, \mathbf{n}+\mathbf{e}_i)$, $(i = a, b)$ on the edges of the lattice.

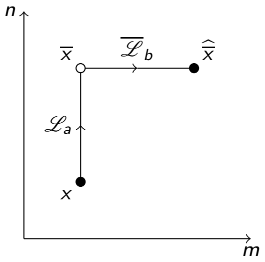


The closure relation $\square\mathscr{L} = 0$, holding on solutions of the equations of motion (comprising corner equations and the 3-point maps), guarantees that the action is independent on elementary variations (flips across quadrilaterals) of the discrete curve Γ (fixing the endpoints).

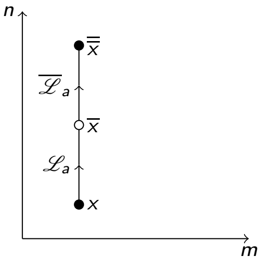
MDC variational principle: action $\mathscr{S}$ must be stationary under variation of the dependent variables, i.e. $x \to x + \delta x$ as well as under variation of the curve $\Gamma \to \Gamma'$ (i.e. varying the independent variables) on the solutions of equation of motion. By choosing different discrete curves we get the compatible set of lattice equations from the 1-form structure.

Elementary Action Configurations

All equations of motion now arise as EL eqns from basic configurations of the curve Γ.



i)



ii)

Action for curve [i)] is given by

$$\begin{aligned}\mathscr{S} &= \mathscr{L}_a(x,\overline{x}) + \mathscr{L}_b(\overline{x},\widehat{\overline{x}}) \\ &= \frac{P-R}{r}[x^2+\overline{x}^2+\tfrac{2}{a}x\overline{x}] + \frac{P-Q}{q}[\overline{x}^2+\widehat{\overline{x}}^2+\tfrac{2}{b}\overline{x\widehat{x}}]\end{aligned}$$

EL equation:

$$\frac{\partial\mathscr{S}}{\partial\overline{x}} = 2\left[\left(\frac{P-R}{r}+\frac{P-Q}{q}\right)\overline{x} + \frac{P+R}{r}x + \frac{P+Q}{q}\widehat{\overline{x}}\right] = 0$$

A corner equation.

Action for curve [ii)] is given by

$$\begin{aligned}\mathscr{S} &= \mathscr{L}_a(x,\overline{x}) + \mathscr{L}_a(\overline{x},\overline{\overline{x}}) \\ &= \frac{P-R}{r}[x^2+\overline{x}^2+\tfrac{2}{a}x\overline{x}] + \frac{P-R}{r}[\overline{x}^2+\overline{\overline{x}}^2+\tfrac{2}{a}\overline{x\overline{x}}]\end{aligned}$$

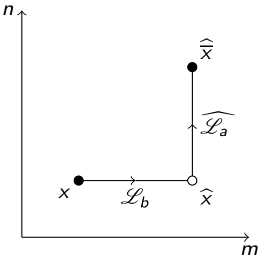
EL equation:

$$\frac{\partial\mathscr{S}}{\partial\overline{x}} = 2\left[2\frac{P-R}{r}\overline{x} + \frac{P+R}{r}\left(x+\overline{\overline{x}}\right)\right] = 0$$

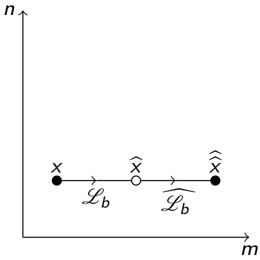
The equation of motion for the "bar" evolution.

Elementary Action Configurations contd.

Similarly, the other corner equation and horizontal map:



iii)



iv)

Action for curve [iii)] is given by

$$\mathscr{S} = \mathscr{L}_b(x,\widehat{x}) + \mathscr{L}_a(\widehat{x},\widehat{\widehat{x}})$$
$$= \frac{P-Q}{q}\left[x^2 + \widehat{x}^2 + \frac{2}{b}x\widehat{x}\right] + \frac{P-R}{r}\left[\widehat{x}^2 + \widehat{\widehat{x}}^2 + \frac{2}{a}\widehat{x}\widehat{\widehat{x}}\right]$$

EL equation:

$$\frac{\partial \mathscr{S}}{\partial \widehat{x}} = 2\left[\left(\frac{P-Q}{q} + \frac{P-R}{r}\right)\widehat{x} + \frac{P+Q}{q}x + \frac{P+R}{r}\widehat{\widehat{x}}\right] = 0$$

The second corner equation.

Action for curve [iv)] is given by

$$\mathscr{S} = \mathscr{L}_b(x,\widehat{x}) + \mathscr{L}_b(\widehat{x},\widehat{\widehat{x}})$$
$$= \frac{P-Q}{q}\left[x^2 + \widehat{x}^2 + \frac{2}{b}x\widehat{x}\right] + \frac{P-Q}{q}\left[\widehat{x}^2 + \widehat{\widehat{x}}^2 + \frac{2}{b}\widehat{x}\widehat{\widehat{x}}\right]$$

EL equation:

$$\frac{\partial \mathscr{S}}{\partial \widehat{x}} = 2\left[2\frac{P-Q}{q}\widehat{x} + \frac{P+Q}{q}\left(x + \widehat{\widehat{x}}\right)\right] = 0$$

The equation of motion for the "hat" evolution.

Quantum 1-form structure: curve-dependent propagators

In the quantum case the Lagrangian 1-form structure leads to the introduction of propagators:

$$K_\Gamma(\partial\Gamma) = \mathscr{N}_\Gamma \left[\prod_{\boldsymbol{\gamma}(\mathbf{n})\in\Gamma} \int_{-\infty}^{\infty} \mathrm{d}x(\mathbf{n}) \right] \; e^{i\mathscr{S}[x(\mathbf{n};\Gamma)]/\hbar} \; .$$

In the quadratic case, using Gaussian integrals, these can be computed explicitly. The following assertions cane be made on the basis of these examples:

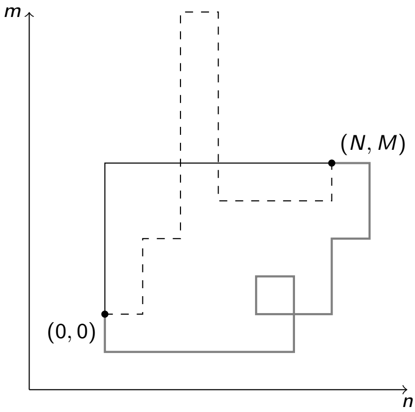
- For Lagrangians obeying classically the closure relation, the propagators are (up to normalisation) independent on the curve, i.e. only depend on the end points;
- conversely the condition on curve-independence fixes the Lagrangian components of the Lagrange 1-form up to a (direction independent) factor and up to an exact 1-form (i.e., functions of $x(\mathbf{n})$ trivially satisfying the closure relation.

These assertions can be verified also by the standard (discrete) time-slicing procedure, starting from a canonical quantization of the classical integrable map.

Quantum variational principle

The analogy between classical and quantum multiform variational principle is as follows:

- *Classical variational principle:* action functional must be critical w.r.t. variations of both the independent variables and the dependent variables (the curve);
- *Quantum variational principle:* (Feynman type) propagators, obtained by summing over all "paths" in terms of the *dependent variable* x, considered as functions of multi-time paths in the independent variable, are stationary w.r.t. the variations in the time-paths Γ.



One way to think of this is to conjecture a novel quantum object: a functional of the Lagrangians, obtained by summing over all time-paths. This object would have critical points at the "admissable" Lagrangians, namely those who possess the above property of stationarity w.r.t. the time-path variations, since it will acquire an infinite number of equal contributions from all deformed paths $\mathfrak{P}$.
The proper definition of such a novel quantum object is under investigation.

HAPPY 60th BIRTHDAY, JEAN-MICHEL!

Conclusions

- We have formulated a quantum variational principle, based on the notion of MDC and Lagrangian multiform structures, in terms of quantum (path integral type) propagators;
- The 1-form structure manifests itself as the (discrete-time) path independence of quantum propagators in multi-time space of independent variables, and was illustrate on the simplest possible example of a multi-time harmonic oscillator;
- The quantum Lagrangian 2-form structure, where the propagators are required to be invariant under deformations of the surface, was illustrated on the basis of the quadratic 3-point case;
- In both cases the dependence on the (lattice) parameters forms a key aspect, while in terms of these one should also consider the continuous path integral;
- The quadratic case (where computations can be performed explicitly in terms of Gaussian integrals) seems to reveal many of the main features of MDC on the quantum level. The aspect that is lost in this case is that of the role of *singularities*;
- Generalization to higher-order cases (coupled harmonic oscillators) and to higher-order linear lattice systems (e.g. coupled quad systems) can be readily done. In the nonlinear (i.e., non-quadratic cases) we expect the integrations to involve more complicated reproducing kernels, (e.g. of Bessel type);
- A formulation in terms of a novel quantum object defined in terms of sum over time-paths, or sums over surfaces, is under investigation (such objects seem also to arise, from a different perspective, in loop quantum gravity, e.g. Reisenberger & Rovelli, 1997).