q-Toda chain

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Collaboration with Olivier Babelon, Karol Kozlowski

En l' honneur de Jean-Michel Maillet.

Toda Classical

Toda classical

- Toda
- Flashka Mac Laughlin
- Kac Van Moerbecke
- Dubrovin Novikov
- Krichever
- $\cdots$

quantization of Toda Chain

Toda quantum

- Gutzwiller
- Gaudin
- Sutherland
- Sklyanin
- Gaudin Pasquier
- Nekrasov Shatashvilii
- Koszlowski Teschner
- $\cdots$

q-Toda

q-Toda quantum

- Hallnas, Ruijsenaars
- Karchev, Lebedv, Semenov Tian Shansky
- Grassi, Hatsuda,Marino
- Faddeev Takhtajan
- Kashaev, Sergeev
- Sciarrapa
- $\cdots$

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This Talk based on a paper in preperation

equation of motion

Toda equations of motion:

$$\frac{d^2 x_i}{dt^2} = e^{x_i - x_{i+1}} - e^{x_{i-1} - x_i}$$

Follow from Hamilton dynamics. H Hamiltonian:

$$H = \sum p_k^2 + \sum e^{x_i - x_{i+1}}$$

connection with XXX

- consider the limit of infinite spin and infinite λ of the XXX Lax matrix, the Gaudin argument goes as follows:

$$L = \begin{pmatrix} u + S^z & \frac{S^-}{\lambda} \\ \frac{S^+}{\lambda} & \frac{u - S^z}{\lambda^2} \end{pmatrix}$$

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- It becomes the Toda Lax Matrix:

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$$L = \begin{pmatrix} u - p & e^q \\ -e^{-q} & 0 \end{pmatrix}$$

- It has no pseudovacuum,
  this is the reason why Sklyanin (who had wisely translated Gaudin book in russian a few years before Jean-Sebastien) got interested in Toda and discovered SOV.

Backlund

- Backlund:

$$W_u = (ux_1 + e^{y_1 - x_1}) - (uy_1 + e^{x_2 - y_1}) + \cdots$$

- Canonical transform

$$p_{x_i} = \frac{\partial W}{\partial x_i} = -u + e^{x_i - y_{i-1}} + e^{y_i - x_i}$$

$$p_{y_i} = -\frac{\partial W}{\partial y_i} = -u + e^{y_i - x_i} + e^{x_{i+1} - y_i}$$

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- self conjugated

$$H_y = H^{\mathrm{toda}}, \;\; H_y = H^{\mathrm{toda}}$$

- Very convenient to construct solitonic solutions (J.S. Gaudin Book).

Backlund and Q operator

- In Quantum Mechanics, H becomes an operator.

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$$Q_u(x_i, y_j) = e^{W_u(x_i, y_j)}$$

- self conjugated $\Leftrightarrow$ $Q(u)$ commutes with H

$$Q_u H = H Q_u$$

- commutation of compositions of canonical transform $\Leftrightarrow$ $Q(u)$ commute at different spectral parameters u and v:

$$Q_u Q_v = Q_v Q_u$$

Backlund and Baxter Q

- $Q(u)$ obeys a difference equation:

$$T(u)Q_u = (-)^N Q_{u-i} + Q_{u+i}$$

with $T(u)$ a degree N polynomial and $\#x_k = N$.

- $T(u)$ is the generating function of the conserved quantities:

$$T(u) = u^N + Pu^{N-1} + Hu^{N-2} + \cdots$$

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- Since all operators commute, TQ equation can be viewed as a scalar equation for the common eigenvales of $T(u)$ and $Q(u)$.
- Can we use this equation to obtain the spectrum of the conserved quantities?

Difficulty with Bethe Ansatz

- From Baxter equation one deduces Bethe equations:

$$\frac{Q(u_k+i)}{Q(u_k-i)} = (-)^{N+1}$$

From this equation one can in principle obtain the Bethe roots u_k and $Q(u)$.

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$$Q(u) \sim \cos N(u \log u - u)$$

- We know $Q(u)$ is entire since the kernel is entire.
- $Q(u) \sim e^{-\frac{N\pi|u|}{2}}$ at large u from WKB analysis.

$N = 1$, Q = Bessel K

- Let us concentrate on the case $N = 1$:
- the Kernel is then a number:

$$Q(u) = \int_{-\infty}^{\infty} e^{iu - 2\cosh(x)} dx$$

- It obeys the difference equation:

$$-2iuQ_u = -Q_{u-i} + Q_{u+i}$$

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- We recognize an integral representation and difference equation satisfied by Bessel $K_{-iu}(1)$.

$N = 1$ and Bessel functions

- The $N = 1$ difference equation admits generically two independent solutions:

$$Q_{\downarrow u} = I_{iu}(1)/\sinh(\pi u)$$
$$Q_{\uparrow u} = I_{-iu}(1)/\sinh(\pi u)$$

- $Q_{\downarrow u}$ and $Q_{\uparrow u}$ have correct assymptotics but poles for $iu =$integer.

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- $Q_{\downarrow u}$ and $Q_{\uparrow u}$ have correct assymptotics but poles for $iu =$integer.
- We obtain the Bessel-K function: $K_u = K_{-iu}(1)$

$$K_u = Q_{\uparrow u} + Q_{\downarrow u}$$

- No poles for iu an integer.
- Exponentially decreasing when $u \to \pm\infty$

N Generic. The old testament: Gutzwiller, Gaudin P

- For N generic, the difference equation rewrites:

$$T(u)Q_u = (-)^N Q_{u-i} + Q_{u+i}$$

with $T(u) = \prod_j -2i(u - v_j)$ a degree N unknown polynomial.

- We obtain two approximate solutions when $iu \to \pm\infty$:

$$Q^0_\uparrow(u; v_j) = (1/2)^{iu} \prod_j \Gamma(\tfrac{u-v_j}{i})$$

$$Q^0_\downarrow(u; v_j) = (1/2)^{-iu} \prod_j \Gamma(\tfrac{v_j-u}{i})$$

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Correct assymptotics but poles.

- We substitute $Q = Q^0 \mu$ into the TQ equation and obtain:

$$\mu_\uparrow(u) = \mu_\uparrow(u + i) + (-)^N \rho \tfrac{\mu_\uparrow(u-i)}{T(u)T(u-i)}$$

$$\mu_\downarrow(u) = \mu_\downarrow(u - i) + (-)^N \rho \tfrac{\mu_\downarrow(u+i)}{T(u)T(u+i)}$$

These are three terms recursion relations for $\mu(u)$, which can be solved (continuous fractions),

- $\mu_\uparrow$ and $\mu_\downarrow$ can be matched. Their Wronskian $W(u)$ is a Hill determinant which must be set equal to zero:

$$W(u) = \prod_j \frac{\sinh \pi(u-u_k)}{\sinh \pi(u-v_k)}$$

u_k are the Bethe roots here defined up to i times an integer.

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  u_k are the Bethe roots here defined up to i times an integer.

- We must require the proportionality coefficiant at Bethe roots to be independant of the Bethe root:

$$\frac{Q_\uparrow}{Q_\downarrow}(u_k) = \xi$$

  These are the Bethe equations.

The new testament: Nekrasov Shatashvili, Kozlowski Teschner

- Introduce back the coupling constant in front of the potential:

$$T(u)Q_u = \rho^{1/2}((-)^N Q_{u-i} + Q_{u+i})$$

The new testament: Nekrasov Shatashvili, Kozlowski Teschner

- Introduce back the coupling constant in front of the potential:

$$T(u)Q_u = \rho^{1/2}((-)^N Q_{u-i} + Q_{u+i})$$

- Order zero Sutherland approximation valid in the limit ρ small:

$$Q^0_\uparrow(u; u_j) = (\rho^{1/2}/2)^{iu} \prod_j \Gamma(\tfrac{u-u_j}{i})$$

$$Q^0_\downarrow(u; u_j) = (\rho^{1/2}/2)^{-iu} \prod_j \Gamma(\tfrac{u_j-u}{i})$$

This important observation is due to Ahn, Fateev, Kim, Rim, Yang.

- Now, Bethe roots instead of v_j appear, one can solve the Bethe equations at this order (Sutherland), called perturbative limit by N.S.

$$(\frac{\rho}{4})^{iu_j} = \prod_k \frac{\Gamma(\frac{u_k - u_j}{i})}{\Gamma(\frac{u_j - u_k}{i})}$$

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$$(\frac{\rho}{4})^{iu_j} = \prod_k \frac{\Gamma(\frac{u_k - u_j}{i})}{\Gamma(\frac{u_j - u_k}{i})}$$

- Call T^0 the solution for T at zero order and substitute $Q = Q^0 \nu$ into the TQ equation:

$$\frac{T}{T_0}\nu_\uparrow(u) = \nu_\uparrow(u+i) + (-)^N \rho \frac{\nu_\uparrow(u-i)}{T_0(u)T_0(u-i)}$$

$$\frac{T}{T_0}\nu_\downarrow(u) = \nu_\downarrow(u-i) + (-)^N \rho \frac{\nu_\downarrow(u+i)}{T_0(u)T_0(u+i)}$$

These are three terms recursion relations for $\nu(u)$, and $T(u)$ which are functions of the seed T_0.

Perturbative solution

- This leads to a systematic expansion of T and ν in powers of ρ which in the $N = 1$ case enables to reconstruct the Bessel function:

$$K_{iu}(\rho^{1/2}) = (Q_{\uparrow u} + Q_{\downarrow u}).$$

- The Bethe equations are obtained as earlier:

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but now at a certain order in ρ and no roots of $T(u)$ are involved in the solution.
- N.S. and K.T. use a slightly different technique obtaining $\nu_\uparrow$ from the Wronskian using a nonlinear integral equation.
- I belive it is equivalent to the perturbation in ρ if one iterates the nonlinear equation starting from $\nu = 1$.
Remark: I impose $\nu_\uparrow(0) = 1$, $\nu_\downarrow(\infty) = 1$.

The q-Toda Chain

- q-Toda (Ruijsenaars) is to Toda what XXZ chain is to XXX chain, or Harper equation is to Mathieu equation. Was revived by Marino and collaborators due to its connection with Toric Calabi-Yau.
$$H = \sum_{k=1}^{N} X_k (1 + \epsilon^2 \frac{x_{k+1}}{x_k})$$

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- Dual system with dual Weyl pair commuting with this one:

$$\tilde{x}\tilde{X} = \tilde{q}\tilde{X}\tilde{x},$$

  with $q = e^{i\omega_1/\omega_2}$, $\tilde{q} = e^{i\omega_2/\omega_1}$.
  Dual Hamiltonian $\tilde{H}$.

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  with $q = e^{i\omega_1/\omega_2}$, $\tilde{q} = e^{i\omega_2/\omega_1}$.
  Dual Hamiltonian $\tilde{H}$.

- q is either of modulus one, in which case H is hermitian. or $\tilde{q} = 1/q^*$ strong coupling case.

Backlund Kernel

- Q has the same strucure as Toda kernel:

  $\psi_1(u) = e_{\omega_1\omega_2}(2iu\sum_{k=0}^{N-1}(q_{2k+1}-q_{2k}))\prod_{k=0}^{2N-1} G_L(q_k-q_{k+1}+\eta-i\frac{\Omega}{4})$

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- but need to be divided by:

$$\psi_2 = \prod_{k=0}^{N-1}\frac{1}{G_L(q_{2k}-q_{2k+2}+2\eta)}$$

- to commute $Q_uQ_v = Q_vQ_u$:

$$Q_u = \psi_1\psi_2$$

- Q is modular invarient: G_L is Faddeev Ruijsenaars Γ function.
- Q is Hilbert Schmidt

- definition:

$$G(z+i\omega_1/2) = 2\cosh(\tfrac{\pi z}{\omega_2})G(z-i\omega_1/2)$$

- integral representation:

$$G(z) = \exp -\tfrac{i}{4}\int_C \tfrac{dt}{t}\frac{e^{2itz}}{\sinh(\omega_1 t)\sinh(\omega_2 t)}$$

Baxter equation

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$$T(u) = \prod_{k=1}^{N} -2i \sinh_{\omega_2}(u - v_k)$$

- Q_u entire function
- Modular invariant
- large u behavior:

$$|Q(u)| \sim_{u\to\pm\infty} e_{\omega_1\omega_2}(-N\Omega|u|/2))$$

Sutherland approximaion

- In the small coupling limit:

$$Q^0_\uparrow(u; v_j) = e_{\omega_1\omega_2}(i2N\eta u)\prod_j \Gamma_q(u - v_j)$$

$$Q^0_\downarrow(u; v_j) = e_{\omega_1\omega_2}(-i2N\eta u)\prod_j \Gamma_q(-u + v_j)$$

where:

$$\rho = e_{\omega_2}(4N\eta)$$

- Modular invariant:

$$\Gamma_q(u) = G(u - i\Omega/2)$$

- Good large $|u|$ behavior, but poles. Their cancellations lead to zero order Bethe equations.

zero order Bethe equations

- at order zero, the Bethe equations read:

$$\frac{\pi}{2} n_k = 2\pi N \frac{u_k \eta}{\omega_1 \omega_2} + \Re \sum_{j \neq k} f(u_k - u_j - i\Omega/2) - \log \xi$$

- where f is obtained from residue evaluation of $\log G$:

$$\Re f(u - i\Omega/2) = \pi\left(\frac{1}{4} + \frac{1}{12}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right) - \frac{u^2}{2\omega_1\omega_2}\right) -$$

$$- \Im \sum_{k \geq 0} \frac{1}{2k}\left(\frac{1+q^k}{1-q^k} e_{\omega_2}(-2ku) + \frac{1+\tilde{q}^k}{1-\tilde{q}^k} e_{\omega_1}(-2ku)\right)$$

dressing

- As for Toda, dress Q^0 with a product ν times $\tilde{\nu}$ to preserve modular invariance.

$$Q_\uparrow = Q^0_\uparrow \nu_\uparrow \tilde{\nu}\uparrow$$

Bethe equations read:

$$B_k(u_j) = B^0_k(u_j) + \frac{1}{2}\Im(\log(R(u_k) + \log(\tilde{R}(u_k))$$

where $R = \nu_\uparrow / \nu_\downarrow$.

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  where $R = \nu_\uparrow / \nu_\downarrow$.

- These equations admit real roots solutions rely on $|R\tilde{R}| = 1$ which is experimentally verifyed in the cases of interest.
  Remark: I impose $\nu_\uparrow(0) = 1$, $\nu_\downarrow(\infty) = 1$.
- For N=2, this equation coincides with a conjecture of Sciarappa if we identify $\nu_\uparrow$ with his "Type II defect instanton partition function" $\hat{Z}^{(c),\mathrm{inst}}_{3d/5d,\mathrm{NS}}$.

N=1

- In the $N = 1$ case, we have $(u = i\omega_1 n)$:

$$Q_\downarrow(u) = (-)^n \frac{q^{\frac{n(n+1)}{2}}}{\theta} I_n^{(2)}(2i\rho^{1/2}q^{1/4})$$

- where

$$\theta = (q^{-n}, q)_\infty (q^{n+1}, q)_\infty (q, q)_\infty \text{ is the elliptic } \theta_3$$

- $I_n^{(2)}$ is the second Jackson q-Bessel function.

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- conjecture

$$\int_{-\infty}^{\infty} e_{\omega_1\omega_2}(2iut - iu^2/2)) G_L(t + \eta - i\Omega/4) G_L(-t + \eta - i\Omega/4) dt$$

$$= \frac{I_n \tilde{I}_n + I_{-n} \tilde{I}_{-n}}{\theta\tilde{\theta}}$$

N=2, comparison with Kashaev-Sergeev

Table: $\omega_1/\omega_2 = i, \eta = 0$

n	root	H
0	.35355339059327376220	4.59435880983691893383 i
-1	.61211737164616726750	-13.87830477803669504 2+6.16129624324434
-2	.79079992732462774105	-31.32504489967257869 9-12.15333894226 76
-3	.93544753055190792792 7	-33.71547676874086490 9-54.17105674969 1

N=2 comparison with Sciarappa

Table: $\omega_1/\omega_2 = 2^{-1/2}, \eta = 0$

n	root	H
0	.462871608964	2.460524271907
-1	.680791907983	3.598470877254
-2	.844632649750	4.4628893132238

Table: $\omega_1 = 2^{-1/2}, \omega_2 = 1, \eta = \log(3)/8\pi$

n	root	H
0	.4354731597837	2.752848101914
-1	.6178613438775	3.883834678235
-2	.7553058969907	4.746028853867

- Quizz. Qui a dit:
  POUR MOI, JE TIENS QUE HORS DE PARIS, IL N'EST POINT DE SALUT POUR LES HONNETES GENS.?
- BON ANNIVERSAIRE JEAN-MICHEL, ET LONGUE VIE A LYON