Local Hamiltonians associated to cyclic representations of the reflection algebra

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Cyclic representations of the reflection algebra

Let $\mathcal{R} \in \mathcal{M}_4(\mathbb{C})$ be the **6-vertex trigonometric R-matrix**. The Yang-Baxter algebra is : $\mathcal{R}_{12}(\lambda/\mu)M_{1Q}(\lambda)M_{2Q}(\mu) = M_{2Q}(\mu)M_{Q1}(\lambda)\mathcal{R}_{12}(\lambda/\mu)$
We consider $L_{0n}(\lambda|P_n)$, the following general cyclic representation [1] of the Yang Baxter algebra :

$$L_{0n}(\lambda|P_n) = \begin{pmatrix} \lambda\alpha_n\mathfrak{v}_n - \beta_n/\lambda\ \mathfrak{v}_n^{-1} & \mathfrak{u}_n\left(q^{-1/2}a_n\mathfrak{v}_n + q^{1/2}b_n\mathfrak{v}_n^{-1}\right) \\ \mathfrak{u}_n^{-1}\left(q^{1/2}c_n\mathfrak{v}_n + q^{-1/2}d_n\mathfrak{v}_n^{-1}\right) & \gamma_n/\lambda\ \mathfrak{v}_n - \lambda\delta_n\mathfrak{v}_n^{-1} \end{pmatrix} \quad \begin{bmatrix} \mathfrak{u}_n\mathfrak{v}_m = q^{\delta_{nm}}\mathfrak{v}_m\mathfrak{u}_n \\ \mathfrak{u}_n^p = \mathfrak{v}_n^p = \mathbb{1} \\ q^p = 1 \end{bmatrix}$$

$\mathfrak{u}_n$ and $\mathfrak{v}_n$ are the generators of a local cyclic Weyl algebra. $dim(\mathcal{H}_n) = p$

One can describe **the sine-Gordon model, the chiral-Potts model or the XXZ spin s chain, with** $p = 2s+1$

To describe general integrable boundaries, we have to consider **the reflection algebra [2]** :

$$\mathcal{R}_{12}(\lambda/\mu)\ \mathcal{U}_{-1Q}(\lambda)\ \mathcal{R}_{12}(\lambda\mu/q)\ \mathcal{U}_{-2Q}(\mu) = \mathcal{U}_{-2Q}(\mu)\ \mathcal{R}_{12}(\lambda\mu/q)\ \mathcal{U}_{-1Q}(\lambda)\ \mathcal{R}_{12}(\lambda/\mu)$$

A solution is given by $\mathcal{U}_{-0Q}(\lambda) = M_{0Q}(\lambda)K_{-0}(\lambda)M_{0Q}^{-1}(1/\lambda)$. The **transfer matrix** reads $T(\lambda|P_Q) = tr_0\left\{K_+(\lambda)\ \mathcal{U}_{-0Q}(\lambda)\right\}$.

In [3] and [4], we solved the spectral problem associated to this transfer matrix using the **Separation of Variables** [5]. For general parameters, the eigenvalues $t(\lambda)$ are a particular type of polynomials, satisfying an **inhomogeneous Baxter equation** :

$$t(\lambda)Q(\lambda) = a(\lambda)Q(\lambda/q) + a(1/\lambda)Q(q\lambda) + F(\lambda)$$

Reflection algebra associated to the cyclic-cyclic fundamental R-matrix

We are interested in the cyclic-cyclic fundamental R-matrix, which does the intertwinning for two different quantum spaces a and b :

$$\mathcal{S}_{ba}(P_b|P_a)\ L_{0a}(\lambda|P_a)\ L_{0b}(\lambda|P_b) = L_{0b}(\lambda|P_b)\ L_{0a}(\lambda|P_a)\ \mathcal{S}_{ba}(P_b|P_a)$$

Existence of $\mathcal{S}$: → The intertwinner is known for the chiral-Potts case. [1]
The parameters have to be on the so-called **chiral-Potts curves**.
→ **We generalized the existence conditions and the expression of $\mathcal{S}$**

The reflection equation for mixed representations :

$$L_{a0}^{\sigma_0\theta_a}(P_a|\lambda)\ K_{-0}(\lambda)\ L_{0a}^{\theta_a}(\lambda|P_a)\ K_{-a}(P_a) = K_{-a}(P_a)\ L_{a0}^{\sigma_0}(\lambda|P_a)\ K_{-0}(\lambda)\ L_{0a}(\lambda|P_a)$$

→ We find an expression for θ. We find a diagonal **scalar solution** $K_{-a}(P_a)$.

Based on [6], we construct the **dual equation** and we find a diagonal scalar solution $K_{+a}(P_a)$.

The dressing of this equation leads to the commutation $[T(\lambda|P_Q), \mathcal{T}(P_a|P_Q)] = 0$,

with the following **cyclic-cyclic fundamental transfer matrix** :

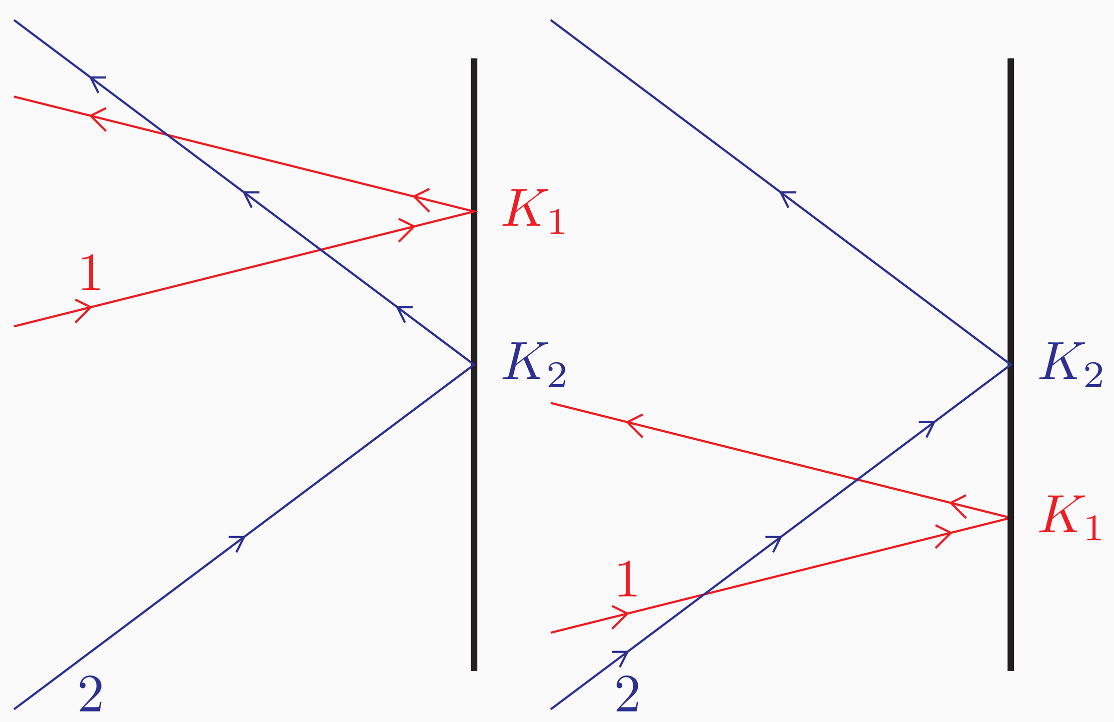
$$\mathcal{T}(P_a|P_Q) = tr_a\left\{K_{+a}(P_a)\ S_{Qa}^{\theta_Q\theta_a}(P_Q|P_a)\ K_{-a}(P_a)\ S_{aQ}^{\theta_Q}(P_a|P_Q)\right\}$$

$S_{Qa}(P_Q|P_a) = S_{Q_Na}(P_{Q_N}|P_a)...S_{Q_1a}(P_{Q_1}|P_a)$ and $S_{aQ}(P_a|P_Q) = S_{aQ_1}(P_a|P_{Q_1})...S_{aQ_N}(P_a|P_{Q_N})$

The cyclic-cyclic reflection equation gives the commutation $[\mathcal{T}(P_b|P_Q), \mathcal{T}(P_a|P_Q)] = 0$.

$$\mathcal{S}_{ba}^{\theta_b\theta_a}(P_b|P_a)\ K_{-a}(P_a)\ \mathcal{S}_{ab}^{\theta_b}(P_a|P_b)\ K_{-b}(P_b) = K_{-b}(P_b)\ \mathcal{S}_{ba}^{\theta_a}(P_b|P_a)\ K_{-a}(P_a)\ \mathcal{S}_{ab}(P_a|P_b)$$

The reflection equations



$$\mathbb{R}_{21}^{\phi_2\phi_1}(P_2|P_1)\ K_{-1}(P_1)\ \mathbb{R}_{12}^{\phi_2}(P_1|P_2)\ K_{-2}(P_2)$$
$$= K_{-2}(P_2)\ \mathbb{R}_{21}^{\phi_1}(P_2|P_1)\ K_{-1}(P_1)\ \mathbb{R}_{12}(P_1|P_2)$$

- $\mathbb{R}$ encodes the collision of the particles, depending on their relative positions at asymptotic times
- The reflected particle is modified by an automorphism ϕ
- The equation holds from asymptotic conservation of momentum

Local Hamiltonians for cyclic models

For cyclic representations, $tr_a\left\{K_{+a}(P_a^-)\right\} = 0$. We thus compute the second order derivative

$$\left.\frac{d^2\ \mathcal{T}(P_a|P_Q)}{d\ x_a^2}\right|_{P_a^-} = \alpha\sum_{k=1}^{N-1}H_{k,k+1} + \alpha\left.\frac{d\ K_{-1}(P_1)}{d\ x_1}\right|_{P_1^-} + B_N + cst$$

The boundary term B_N has a non trivial expression involving K_+, $\left.\frac{dK_+}{dx}\right|_{P^-}$ and the first and second order derivatives of S.

For a quantum space of dimension 3, we explicitely check :

The hamiltonian boundaries are symmetric and $tr_a\left\{K_{+a}(P_a^-)H_{Na}\right\} \propto id$

Standard local Hamiltonians

The standard procedure introduced by Sklyanin [2] is to consider the first order derivative of the transfer matrix. It leads to :

$$\left.\frac{d\ \mathcal{T}(P_a|P_Q)}{dx_a}\right|_{P_a^-} = tr_a\left\{K_{+a}(P_a^-)\right\}\left.\frac{dK_{-1}(P_1)}{dx_1}\right|_{P_1^-}$$
$$+\ tr_a\left\{K_{+a}(P_a^-)\right\}\sum_{k=1}^{N-1}H_{k,k+1}$$
$$+\ tr_a\left\{K_{+a}(P_a^-)H_{Na}\right\} + cst$$

The local interactions are expressed thanks to the derivative of S and the permutation operators :

$$H_{k,m} = \frac{dS_{mk}(P_m|P_k)}{dx_m}\mathcal{P}_{km} + \mathcal{P}_{km}\frac{dS_{mk}(P_m|P_k)}{dx_m}$$

References

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Perspective

There may hold a symmetry of the transfer matrix to explain the boundary symmetry. This would also prove that $tr_a\left\{K_{+a}(P_a^-)H_{Na}\right\}$ reduces to a scalar. Moreover, we can investigate for non diagonal scalar solutions to the cyclic fundamental reflection equations, and for another automorphism θ.