Scalar products for the Zamolodchikov-Fateev model

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- Fusion
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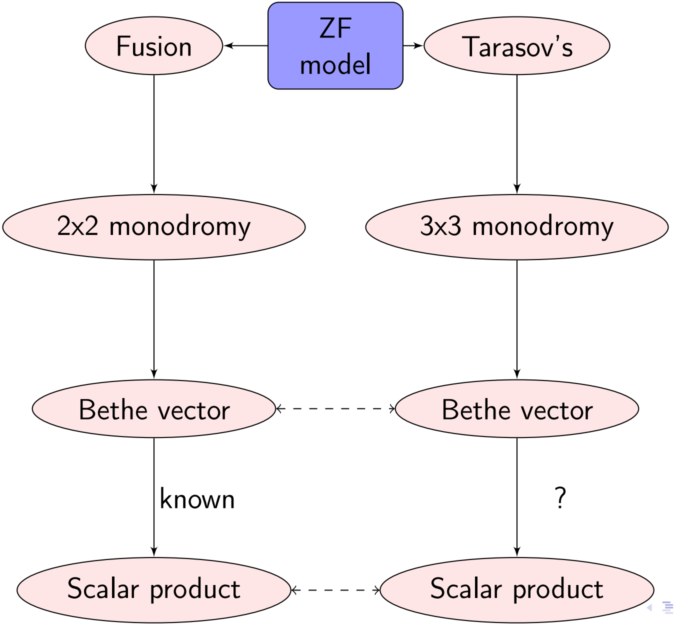
Conclusion

19-vertex models

- 19-vertex models are 3-state vertex models that satisfy the ice-rule.
  - Many different solutions of the Yang-Baxter equation - among them, the Zamolodchikov-Fateev (ZF) [Zamolodchikov-Fateev 80] and the Izergin-Korepin (IK) [Izergin-Korepin 81] models.
  - R-matrix is a 9x9 matrix.
  - Bethe ansatz can be implemented by the Tarasov's construction [Tarasov 88].
  - Here we focus on the ZF model. The objective is to find the Slavnov formula for the Tarasov-Bethe vectors, in terms of "19-vertex variables".

Two Bethe ansätz for ZF



R-matrices and K-matrices

- One way to obtain the ZF model R-matrix and K-matrix is by means of the fusion technique [Kulish-Reshetikhin-Sklyanin 81,Kulish-Sklyanin 82]. We start with the fundamental R-matrix,

$$R^{(\frac{1}{2},\frac{1}{2})}(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}$$

  which acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and where η is a free-parameter.
- R-matrix drops in rank at $u = \eta$, which allows us to define [Gohmann-Seel-Suzuki 10,Beisert-Leeuw-Nag 15]:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F = E^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- E, F are analogous of the standard projectors P, but they have the advantage to give directly the right dimension of the fused matrices.

R-matrices and K-matrices

- We define the fused R-matrix, which acts on $\mathbb{C}^2 \otimes \mathbb{C}^3$:

$$R^{(\frac{1}{2},1)}_{1\langle 23\rangle}(u) = \frac{1}{\sinh(u+\frac{\eta}{2})} F_{\langle 23\rangle}\, R^{(\frac{1}{2},\frac{1}{2})}_{13}(u+\tfrac{\eta}{2})\, R^{(\frac{1}{2},\frac{1}{2})}_{12}(u-\tfrac{\eta}{2})\, E_{\langle 23\rangle}$$

$$= \begin{pmatrix} \sinh(u+\frac{3\eta}{2}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \sinh(u+\frac{\eta}{2}) & 0 & \frac{1}{\sqrt{2}}\sinh(2\eta) & 0 & 0 \\ 0 & 0 & \sinh(u-\frac{\eta}{2}) & 0 & \sqrt{2}\sinh(\eta) & 0 \\ 0 & \sqrt{2}\sinh(\eta) & 0 & \sinh(u-\frac{\eta}{2}) & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}}\sinh(2\eta) & 0 & \sinh(u+\frac{\eta}{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \sinh(u+\frac{3\eta}{2}) \end{pmatrix}$$

- We fuse again to obtain,

$$R^{(1,1)}_{\langle 12\rangle\langle 34\rangle}(u) = F_{\langle 12\rangle}\, R^{(\frac{1}{2},1)}_{1\langle 34\rangle}(u+\tfrac{\eta}{2})\, R^{(\frac{1}{2},1)}_{2\langle 34\rangle}(u-\tfrac{\eta}{2})\, E_{\langle 12\rangle}$$

$$= \begin{pmatrix} a(u) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(u) & 0 & c(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f(u) & 0 & d(u) & 0 & h(u) & 0 & 0 \\ 0 & c(u) & 0 & b(u) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{d}(u) & 0 & e(u) & 0 & \tilde{d}(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(u) & 0 & c(u) & 0 \\ 0 & 0 & h(u) & 0 & d(u) & 0 & f(u) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c(u) & 0 & b(u) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a(u) \end{pmatrix}$$

- Similarly for the K-matrix, we start with the fundamental K^--matrix,

$$K^{-\,(\frac{1}{2})}(u) = \mathrm{diag}\left(\sinh(u+\xi^-)\,, -\sinh(u-\xi^-)\right)$$

and fuse [Mezincescu-Nepomechie-Rittenberg 90,Mezincescu-Nepomechie 92,Nepomechie-RAP 15]:

$$\begin{aligned}
&K^{-\,(1)}(u) \\
&= \frac{1}{2\sinh(u+\frac{\eta}{2})} F_{\langle 12\rangle}\, K_1^{-\,(\frac{1}{2})}(u+\tfrac{\eta}{2})\, R_{12}^{(\frac{1}{2},\frac{1}{2})}(2u)\, K_2^{-\,(\frac{1}{2})}(u-\tfrac{\eta}{2})\, \mathcal{P}_{12}\, E_{\langle 12\rangle} \\
&= \mathrm{diag}\left(k_1^-(u)\,, k_2^-(u)\,, k_3^-(u)\right)
\end{aligned}$$

where ξ^- is an arbitrary boundary parameter.

- For K^+, we similarly take

$$K^{+\,(1)}(u) = K^{-\,(1)}(-u-\rho^{(1,1)})\Big|_{\xi^-\to\xi^+}$$

R-matrices and K-matrices

- We have now the basic building blocks to construct monodromy and transfer matrices.
- The single-row monodromy matrix for the $R^{(\frac{1}{2},1)}$-matrix is defined as,

$$T_a^{(\frac{1}{2},1)}(u) = R_{aN}^{(\frac{1}{2},1)}(u) \ldots R_{a1}^{(\frac{1}{2},1)}(u)$$

and the associated transfer matrix by,

$$t^{(\frac{1}{2},1)}(u) = \mathrm{tr}_a \, T_a^{(\frac{1}{2},1)}(u)$$

- The single-row monodromy matrix for the $R^{(1,1)}$-matrix is defined as,

$$T_a^{(1,1)}(u) = R_{aN}^{(1,1)}(u) \ldots R_{a1}^{(1,1)}(u)$$

and

$$t^{(1,1)}(u) = \mathrm{tr}_a \, T_a^{(1,1)}(u)$$

R-matrices and K-matrices

- The Yang-Baxter equations imply,

$$R_{12}^{(\frac{1}{2},\frac{1}{2})}(u-v)\,T_1^{(\frac{1}{2},1)}(u)\,T_2^{(\frac{1}{2},1)}(v) = T_2^{(\frac{1}{2},1)}(v)\,T_1^{(\frac{1}{2},1)}(u)R_{12}^{(\frac{1}{2},\frac{1}{2})}(u-v)$$

and

$$R_{12}^{(1,1)}(u-v)\,T_1^{(1,1)}(u)\,T_2^{(1,1)}(v) = T_2^{(1,1)}(v)\,T_1^{(1,1)}(u)R_{12}^{(1,1)}(u-v)$$

as well as,

$$\left[t^{(\frac{1}{2},1)}(u)\,,\,t^{(\frac{1}{2},1)}(v)\right]=0\,,\qquad \left[t^{(1,1)}(u)\,,\,t^{(1,1)}(v)\right]=0$$

and

$$\left[t^{(\frac{1}{2},1)}(u)\,,\,t^{(1,1)}(v)\right]=0$$

- The latter relation implies that $t^{(\frac{1}{2},1)}(u)$ and $t^{(1,1)}(v)$ can be diagonalized simultaneously. In addition, one has the important relation

$$T^{(1,1)}_{\langle 12\rangle}(u) = F_{\langle 12\rangle}\, T_1^{(\frac{1}{2},1)}(u+\tfrac{\eta}{2})\, T_2^{(\frac{1}{2},1)}(u-\tfrac{\eta}{2})\, E_{\langle 12\rangle}$$

which will be used to relate the Bethe vectors from $T^{(1,1)}(u)$ and $T^{(\frac{1}{2},1)}(u)$.

R-matrices and K-matrices

- In order to construct double-row objects, one needs to introduce "reflected" single-row monodromy matrices,

$$\widehat{T}_a^{(\frac{1}{2},1)}(u) = R_{a1}^{(\frac{1}{2},1)}(u) \dots R_{aN}^{(\frac{1}{2},1)}(u)$$

and similarly

$$\widehat{T}_a^{(1,1)}(u) = R_{a1}^{(1,1)}(u) \dots R_{aN}^{(1,1)}(u)$$

- The corresponding double-row monodromy matrices are then defined as follows

$$\begin{aligned} U_a^{(\frac{1}{2},1)}(u) &= T_a^{(\frac{1}{2},1)}(u)\, K_a^{-(\frac{1}{2})}(u)\, \widehat{T}_a^{(\frac{1}{2},1)}(u) \\ U_a^{(1,1)}(u) &= T_a^{(1,1)}(u)\, K_a^{-(1)}(u)\, \widehat{T}_a^{(1,1)}(u) \end{aligned}$$

- They obey the boundary Yang-Baxter equations, in particular

$$R_{12}^{(\frac{1}{2},\frac{1}{2})}(u-v)\,U_1^{(\frac{1}{2},1)}(u)\;R_{12}^{(\frac{1}{2},\frac{1}{2})}(u+v)\,U_2^{(\frac{1}{2},1)}(v)$$
$$=U_2^{(\frac{1}{2},1)}(v)\,R_{12}^{(\frac{1}{2},\frac{1}{2})}(u+v)\,U_1^{(\frac{1}{2},1)}(u)\,R_{12}^{(\frac{1}{2},\frac{1}{2})}(u-v)$$

and

$$R_{12}^{(1,1)}(u-v)\,U_1^{(1,1)}(u)\;R_{12}^{(1,1)}(u+v)\,U_2^{(1,1)}(v)$$
$$=U_2^{(1,1)}(v)\,R_{12}^{(1,1)}(u+v)\,U_1^{(1,1)}(u)\,R_{12}^{(1,1)}(u-v)$$

- Analogous to the single-row case, they are related by

$$U_{\langle 12\rangle}^{(1,1)}(u)$$
$$=\frac{1}{2\sinh(u+\frac{\eta}{2})}F_{\langle 12\rangle}\,U_1^{(\frac{1}{2},1)}(u+\tfrac{\eta}{2})\,R_{12}^{(\frac{1}{2},\frac{1}{2})}(2u)\,U_2^{(\frac{1}{2},1)}(u-\tfrac{\eta}{2})\,\mathcal{P}_{12}\,E_{\langle 12\rangle}$$

- Finally, we define the dpuble-row transfer matrices,

$$\tau^{(\frac{1}{2},1)}(u) = \mathrm{tr}_a\, K_a^{+(\frac{1}{2})}(u)\, U_a^{(\frac{1}{2},1)}(u)$$

$$\tau^{(1,1)}(u) = \mathrm{tr}_a\, K_a^{+(1)}(u)\, U_a^{(1,1)}(u)$$

- These transfer matrices obey

$$\left[\tau^{(\frac{1}{2},1)}(u)\,,\tau^{(\frac{1}{2},1)}(v)\right] = 0\,, \qquad \left[\tau^{(1,1)}(u)\,,\tau^{(1,1)}(v)\right] = 0$$

as well as

$$\left[\tau^{(\frac{1}{2},1)}(u)\,,\tau^{(1,1)}(v)\right] = 0$$

- We use the 2-dimensional auxiliary space representation, *i.e.*, we set

$$T_a^{\left(\frac{1}{2},1\right)}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a$$

in which each entry is an operator acting on the vector space $\left(\mathbb{C}^3\right)^{\otimes N}$.

- We also introduce the reference state vector

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^{\otimes N}$$

as well as its dual,

$$\langle 0| = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^{\otimes N}$$

such that $\langle 0|0\rangle = 1$.

Bethe ansatz: single-row transfer matrix - fusion

- The action of the monodromy operators on the reference state are given by,

$$A(u)|0\rangle = \lambda_1(u)|0\rangle\,, \quad D(u)|0\rangle = \lambda_2(u)|0\rangle\,, \quad C(u)|0\rangle = 0$$

$$\langle 0|A(u) = \langle 0|\lambda_1(u)\,, \quad \langle 0|D(u) = \langle 0|\lambda_2(u)\,, \quad \langle 0|B(u) = 0$$

where

$$\lambda_1(u) = \sinh\left(u + \frac{3\eta}{2}\right)^N\,, \quad \lambda_2(u) = \sinh\left(u - \frac{\eta}{2}\right)^N$$

- The Bethe vectors are given by,

$$|\phi_m(u_1, \ldots, u_m)\rangle = B(u_1)\ldots B(u_m)|0\rangle$$

and

$$\langle\phi_m(u_1, \ldots, u_m)| = \langle 0|C(u_1)\ldots C(u_m)$$

and they satisfy,

$$t^{(\frac{1}{2},1)}(u)|\phi_m(u_1, \ldots, u_m)\rangle = \lambda(u, u_1, \ldots u_m)|\phi_m(u_1, \ldots, u_m)\rangle$$
$$+\sum_{j=1}^{m} \sinh(\eta) F(u, u_j)\frac{E_j(u_j)}{Q_j(u_j)} B(u)|\phi_{m-1}(u_1, \ldots, \hat{u}_j, \ldots, u_m)\rangle$$

and

$$\langle\phi_m(u_1, \ldots, u_m)|t^{(\frac{1}{2},1)}(u) = \langle\phi_m(u_1, \ldots, u_m)|\lambda(u, u_1, \ldots u_m)$$
$$+\sum_{j=1}^{m} \langle\phi_m(u_1, \ldots, \hat{u}_j, \ldots, u_m)|C(u)\frac{E_j(u_j)}{Q_j(u_j)} F(u, u_j)\sinh(\eta)$$

where

Bethe ansatz: single-row transfer matrix - fusion

$$\lambda(u, u_1, \dots u_m) = \lambda_1(u)\frac{Q(u-\eta)}{Q(u)} + \lambda_2(u)\frac{Q(u+\eta)}{Q(u)}$$

is the eigenvalue of the transfer matrix with

$$E_j(u) = \lambda_1(u)Q_j(u-\eta) - \lambda_2(u)Q_j(u+\eta)$$

being the Bethe ansatz polynomial, and

$$F(u,v) = \frac{1}{\sinh(u-v)}$$

In the above formulae we have also introduced the Baxter Q-polynomial,

$$Q(u) = \prod_{i=1}^{m} \sinh(u-u_i)$$

as well as the indexed Baxter Q-polynomial,

$$Q_j(u) = \prod_{i\neq j}^{m} \sinh(u-u_i)$$

Bethe ansatz: single-row transfer matrix - fusion

- Finally, let us suppose that $\{u_1, \ldots, u_m\}$ satisfy the Bethe equations (*i.e.*, they are on-shell rapidities) and that there is no restriction on $\{v_1, \ldots, v_m\}$ (*i.e.*, they are off-shell rapidities). Then, the scalar product $\langle \phi_m(u_1, \ldots, u_m) | \phi_m(v_1, \ldots, v_m) \rangle$ is given by the determinant formula [Slavnov 89]

$$\langle \phi_m(u_1, \ldots, u_m) | \phi_m(v_1, \ldots, v_m) \rangle = \prod_{i=1}^{m} \lambda_2(u_i) \frac{\det_m \left(\frac{\partial}{\partial u_i} \lambda(v_j, u_1, \ldots, u_m) \right)}{\det_m \left(F(v_i, u_j) \right)}$$

- Using

$$\frac{\partial}{\partial u_i} \lambda(v_j, u_1, \ldots, u_m) = \sinh(\eta) F(u_i, v_j) F(v_j, u_i) \frac{E_i(v_j)}{Q_i(v_j)}$$

we can rewrite the Slavnov formula as,

$$\langle \phi_m(u_1, \ldots, u_m) | \phi_m(v_1, \ldots, v_m) \rangle = \prod_{i=1}^{m} \lambda_2(u_i) \frac{\det_m \left(\sinh(\eta) F(u_i, v_j) F(v_j, u_i) \frac{E_i(v_j)}{Q_i(v_j)} \right)}{\det_m \left(F(v_i, u_j) \right)}$$

Bethe ansatz: single-row transfer matrix - Tarasov's construction

- We now consider the solution of the ZF model within Tarasov's construction [Lima-Santos 99]. Here, we use a 3-dimensional auxiliary space, i.e.,

$$T_a^{(1,1)}(u) = \begin{pmatrix} A_1(u) & B_1(u) & B_2(u) \\ C_1(u) & A_2(u) & B_3(u) \\ C_2(u) & C_3(u) & A_3(u) \end{pmatrix}_a$$

- Here we have the action,

$$A_j(u)|0\rangle = \Lambda_j(u)|0\rangle \,, \quad C_j(u)|0\rangle = 0$$

$$\langle 0|A_j(u) = \langle 0|\Lambda_j(u) \,, \quad \langle 0|B_j(u) = 0$$

for $j = 1, 2, 3$ and where

$$\Lambda_1(u) = (\sinh(u+\eta)\sinh(u+2\eta))^N \,,$$
$$\Lambda_2(u) = (\sinh(u)\sinh(u+\eta))^N \,,$$
$$\Lambda_3(u) = (\sinh(u)\sinh(u-\eta))^N$$

Bethe ansatz: single-row transfer matrix - Tarasov's construction

- The Bethe vector is constructed by means of the recursion relation,

$$|\psi_m(u_1,\dots,u_m)\rangle = B_1(u_1)|\psi_{m-1}(u_2,\dots,u_m)\rangle$$
$$-B_2(u_1)\sum_{i=2}^{m}\gamma_i^{(m)}(u_1,\dots,u_m)|\psi_{m-2}(u_2,\dots,\hat{u}_i,\dots,u_m)\rangle$$

where

$$\gamma_i^{(m)}(u_1,\dots,u_m)$$
$$= 2\sinh(\eta)\Lambda_1(u_i)F(u_1,u_i+\eta)\frac{Q_{1,i}(u_i-2\eta)}{Q_{1,i}(u_i)}\prod_{j=2,j<i}^{m}\Omega(u_i,u_j)$$

and

$$\Omega(u,v)=\frac{\sinh(u-v-\eta)\sinh(u-v+2\eta)}{\sinh(u-v-2\eta)\sinh(u-v+\eta)}$$

Bethe ansatz: single-row transfer matrix - Tarasov's construction

- The dual Bethe vector is given by,

$$\begin{aligned} \langle \psi_m(u_1, \ldots, u_m)| &= \langle \psi_{m-1}(u_2, \ldots, u_m)| C_1(u_1) \\ - & \sum_{i=2}^{m} \tilde{\gamma}_i^{(m)}(u_1, \ldots, u_m) \langle \psi_{m-2}(u_2, \ldots, \hat{u}_i, \ldots, u_m)| C_2(u_1) \end{aligned}$$

where

$$\begin{aligned} &\tilde{\gamma}_i^{(m)}(u_1, \ldots, u_m) \\ &= \sinh(2\eta) \cosh(\eta) \Lambda_1(u_i) F(u_1, u_i + \eta) \frac{Q_{1,i}(u_i - 2\eta)}{Q_{1,i}(u_i)} \prod_{j=2, j<i}^{m} \Omega(u_i, u_j) \end{aligned}$$

Notice that the the initial conditions

$$|\psi_0\rangle = |0\rangle \,, \quad \langle \psi_0| = \langle 0|$$

are assumed.

Bethe ansatz: single-row transfer matrix - Tarasov's construction

- The off-shell equation in this case in more intricate:

$$
\begin{aligned}
& t^{(\mathbf{1},\mathbf{1})}(u)|\psi_m(u_{\mathbf{1}},\dots,u_m)\rangle = \Lambda(u,u_{\mathbf{1}},\dots,u_m)|\psi_m(u_{\mathbf{1}},\dots,u_m)\rangle \\
& + \sum_{j=\mathbf{1}}^{m} \sinh(2\eta)F(u,u_j)\frac{\bar{E}_j(u_j)Q_j(u_j-2\eta)}{Q_j(u_j)Q_j(u_j-\eta)}\prod_{p<j}^{m}\Omega(u_j,u_p)B_{\mathbf{1}}(u)|\psi_{m-\mathbf{1}}(u_{\mathbf{1}},\dots,\hat{u}_j,\dots,u_m)\rangle \\
& + \sum_{j=\mathbf{1}}^{m} 2\sinh(\eta)F(u,u_j+\eta)\frac{\bar{E}_j(u_j)Q_j(u_j-2\eta)}{Q_j(u_j)Q_j(u_j-\eta)}\prod_{p<j}^{m}\Omega(u_j,u_p)B_{\mathbf{3}}(u)|\psi_{m-\mathbf{1}}(u_{\mathbf{1}},\dots,\hat{u}_j,\dots,u_m)\rangle \\
& + \sum_{j<k}^{m} H_{jk}^{(m)}(u,u_{\mathbf{1}},\dots,u_m)B_{\mathbf{2}}(u)|\psi_{m-\mathbf{2}}(u_{\mathbf{1}},\dots,\hat{u}_j,\dots,\hat{u}_k,\dots,u_m)\rangle
\end{aligned}
$$

where

$$
\bar{E}_j(u) = \Lambda_{\mathbf{1}}(u)Q_j(u-\eta) - \Lambda_{\mathbf{2}}(u)Q_j(u+\eta)
$$

and similarly for the dual Bethe vector. The off-shell equation is a consequence of the Yang-Baxter algebra for the 3×3 monodromy matrix in the Tarasov's construction.

Bethe ansatz: single-row transfer matrix - Tarasov's construction

- Computing scalar products using the Yang-Baxter algebra for $R^{(1,1)}$ is thus a hard task. The idea is then to use the equation

$$T^{(1,1)}_{\langle 12\rangle}(u) = F_{\langle 12\rangle}\, T_1^{(\frac{1}{2},1)}(u+\tfrac{\eta}{2})\, T_2^{(\frac{1}{2},1)}(u-\tfrac{\eta}{2})\, E_{\langle 12\rangle}$$

to relate $|\psi_m(u_1,\dots,u_m)\rangle$ and $|\phi_m(u_1,\dots,u_m)\rangle$. That is, we can express the 3x3 monodromy operators in terms of the 2x2 monodromy operators. For example, we have,

$$\begin{aligned}
A_1(u) &= A\left(u+\frac{\eta}{2}\right) A\left(u-\frac{\eta}{2}\right)\,, \\
A_2(u) &= \frac{1}{2}\left(A\left(u+\frac{\eta}{2}\right) D\left(u-\frac{\eta}{2}\right) + D\left(u+\frac{\eta}{2}\right) A\left(u-\frac{\eta}{2}\right)\right. \\
&\qquad \left. + B\left(u+\frac{\eta}{2}\right) C\left(u-\frac{\eta}{2}\right) + C\left(u+\frac{\eta}{2}\right) B\left(u-\frac{\eta}{2}\right)\right)
\end{aligned}$$

etc.

Bethe ansatz: single-row transfer matrix - Tarasov's construction

- Using the above expression, in addition to identities between $\lambda_i(u)$ and $\Lambda_i(u)$, we find that,

$$|\psi_m(u_1,\ldots,u_m)\rangle = s(u_1,\ldots,u_m)\left|\phi_m\left(u_1+\frac{\eta}{2},\ldots,u_m+\frac{\eta}{2}\right)\right\rangle$$

and

$$\langle\psi_m(u_1,\ldots,u_m)| = \left\langle\phi_m\left(u_1+\frac{\eta}{2},\ldots,u_m+\frac{\eta}{2}\right)\right|s(u_1,\ldots,u_m)\cosh(\eta)^m$$

where

$$s(u_1,\ldots,u_m) = 2^{\frac{m}{2}}\prod_{i=1}^{m}\lambda_1\left(u_i-\frac{\eta}{2}\right)\prod_{j<k}^{m}\frac{\sinh(u_j-u_k-2\eta)}{\sinh(u_j-u_k-\eta)}$$

Bethe ansatz: single-row transfer matrix - Tarasov's construction

- The Slavnov scalar product is now easy to compute:

$$\langle\psi_m(u_1,\ldots,u_m)|\psi_m(v_1,\ldots,v_m)\rangle = \cosh(\eta)^m s(u_1,\ldots,u_m)s(v_1,\ldots,v_m)$$
$$\times\prod_{i=1}^{m}\lambda_2\left(u_i+\frac{\eta}{2}\right)\frac{\det_m\left(\frac{\partial}{\partial u_i}\lambda\left(v_j+\frac{\eta}{2},u_1+\frac{\eta}{2},\ldots,u_m+\frac{\eta}{2}\right)\right)}{\det_m\left(F(v_i,u_j)\right)},$$

or, in terms of "19-vertex variables",

$$\begin{aligned}&\langle\psi_m(u_1,\ldots,u_m)|\psi_m(v_1,\ldots,v_m)\rangle\\&=\prod_{j<k}^{m}\frac{\sinh(u_j-u_k-2\eta)}{\sinh(u_j-u_k-\eta)}\frac{\sinh(v_j-v_k-2\eta)}{\sinh(v_j-v_k-\eta)}\\&\times\prod_{i=1}^{m}\Lambda_2(u_i)\frac{\det_m\left(\sinh(2\eta)F(u_i,v_j)F(v_j,u_i)\frac{\bar{E}_i(v_j)}{Q_i(v_j)}\right)}{\det_m\left(F(v_i,u_j)\right)}\end{aligned}\tag{2.1}$$

i.e., in terms of the Bethe ansatz polynomial and the Baxter polynomial.

- We set [Sklyanin 88]

$$U_a^{(\frac{1}{2},1)}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) + \frac{\sinh(\eta)}{\sinh(2u+\eta)}\mathcal{A}(u) \end{pmatrix}_a$$

- Action of the double-row monodromy operators on the reference state:

$$\mathcal{A}(u)|0\rangle = \delta_1(u)|0\rangle\,, \quad \mathcal{D}(u)|0\rangle = \delta_2(u)|0\rangle\,, \quad \mathcal{C}(u)|0\rangle = 0$$

$$\langle 0|\mathcal{A}(u) = \langle 0|\delta_1(u)\,, \quad \langle 0|\mathcal{D}(u) = \langle 0|\delta_2(u)\,, \quad \langle 0|\mathcal{B}(u) = 0$$

where

$$\delta_1(u) = \sinh(u+\xi^-)\sinh\left(u+\frac{3\eta}{2}\right)^{2N}\,,$$

$$\delta_2(u) = \frac{\sinh(2u)\sinh(\xi^- - u - \eta)}{\sinh(2u+\eta)}\sinh\left(u-\frac{\eta}{2}\right)^{2N}$$

Bethe ansatz: double-row transfer matrix - fusion

- The Bethe vectors are given by,

$$|\Phi_m(u_1,\ldots,u_m)\rangle = \mathcal{B}(u_1)\ldots\mathcal{B}(u_m)|0\rangle$$

and

$$\langle\Phi_m(u_1,\ldots,u_m)| = \langle 0|\mathcal{C}(u_1)\ldots\mathcal{C}(u_m)$$

- Off-shell equation:

$$\tau^{\left(\frac{1}{2},1\right)}(u)|\Phi_m(u_1,\ldots,u_m)\rangle = \delta(u,u_1,\ldots,u_m)|\Phi_m(u_1,\ldots,u_m)\rangle$$
$$+\sum_{j=1}^{m}\sinh(\eta)\sinh(2(u+\eta))\frac{\mathcal{F}\,(u,u_j)}{\sinh(2u_j+\eta)}\frac{\mathcal{E}_j(u_j)}{\mathcal{Q}_j(u_j)}\mathcal{B}(u)|\Phi_{m-1}(u_1,\ldots,\hat{u}_j,\ldots,u_m)\rangle$$

$$\langle\Phi_m(u_1,\ldots,u_m)|\tau^{\left(\frac{1}{2},1\right)}(u) = \langle\Phi_m(u_1,\ldots,u_m)|\delta(u,u_1,\ldots u_m)$$
$$+\sum_{j=1}^{m}\langle\Phi_m(u_1,\ldots,\hat{u}_j,\ldots,u_m)|\mathcal{C}(u)\frac{\mathcal{E}_j(u_j)}{\mathcal{Q}_j(u_j)}\frac{\mathcal{F}\,(u,u_j)}{\sinh(2u_j+\eta)}\sinh(2(u+\eta))\sinh(\eta)$$

- We have now "doubled" quantities,

$$\begin{aligned}\mathcal{E}_j(u) &= \sinh(2u)\sinh(u-\xi^+)\delta_1(u)\mathcal{Q}_j(u-\eta)\\ &+ \sinh(2u+\eta)\sinh(u+\eta+\xi^+)\delta_2(u)\mathcal{Q}_j(u+\eta)\end{aligned}$$

$$\mathcal{F}(u,v) = \frac{1}{\sinh(u-v)\sinh(u+v+\eta)}$$

and the double-row Baxter Q-polynomial,

$$\mathcal{Q}(u) = \prod_{i=1}^{m}\sinh(u-u_i)\sinh(u+u_i+\eta)$$

$$\mathcal{Q}_j(u) = \prod_{i\neq j}^{m}\sinh(u-u_i)\sinh(u+u_i+\eta)$$

- The eigenvalue of the transfer matrix:

$$\begin{aligned}\delta(u,u_{\mathbf{1}},\ldots,u_m) &= \frac{\sinh(2(u+\eta))\sinh(u-\xi^+)}{\sinh(2u+\eta)}\delta_{\mathbf{1}}(u)\frac{\mathcal{Q}(u-\eta)}{\mathcal{Q}(u)}\\ &- \sinh(u+\eta+\xi^+)\delta_{\mathbf{2}}(u)\frac{\mathcal{Q}(u+\eta)}{\mathcal{Q}(u)}\end{aligned}$$

- Slavnov scalar product [Kitanine-Kozlowski-Maillet-Niccoli-Slavnov-Terras 07]:

$$\langle \Phi_m(u_1,\ldots,u_m)|\Phi_m(v_1,\ldots,v_m)\rangle = \prod_{i=1}^{m} \frac{\delta_2(u_i)}{\sinh(2(v_i+\eta))\sinh(u_i-\xi^+)} \prod_{j<i}^{m} \frac{\sinh(u_i+u_j+2\eta)}{\sinh(u_i+u_j)} \times \frac{\det_m\left(\frac{\partial}{\partial u_i}\delta(v_j,u_1,\ldots,u_m)\right)}{\det_m\left(\mathcal{F}(v_i,u_j)\right)}$$

or

$$\langle \Phi_m(u_1,\ldots,u_m)|\Phi_m(v_1,\ldots,v_m)\rangle = \prod_{i=1}^{m} \frac{\sinh(2u_i+\eta)\delta_2(u_i)}{\sinh(2v_i+\eta)\sinh(u_i-\xi^+)} \prod_{j<i}^{m} \frac{\sinh(u_i+u_j+2\eta)}{\sinh(u_i+u_j)} \times \frac{\det_m\left(\sinh(\eta)\mathcal{F}(u_i,v_j)\mathcal{F}(v_j,u_i)\frac{\mathcal{E}_i(v_j)}{\mathcal{Q}_i(v_j)}\right)}{\det_m\left(\mathcal{F}(v_i,u_j)\right)}$$

Bethe ansatz: double-row transfer matrix - Tarasov's construction

- The 3-dimensional auxiliary space representation is [Fan 97, Kurak-Lima-Santos 04],

$$U_a^{(1,1)}(u) = \begin{pmatrix} \mathcal{A}_1(u) & \mathcal{B}_1(u) & \mathcal{B}_2(u) \\ \mathcal{C}_1(u) & \mathcal{A}_2(u) + \frac{\sinh(2\eta)}{\sinh(2(u+\eta))}\mathcal{A}_1(u) & \mathcal{B}_3(u) \\ \mathcal{C}_2(u) & \mathcal{C}_3(u) & \mathcal{A}_3(u) + \frac{\sinh(\eta)\sinh(2\eta)}{\sinh(2(u+\eta))\sinh(2u+\eta)}\mathcal{A}_1(u) + \frac{\sinh(2\eta)}{\sinh(2u)}\mathcal{A}_2(u) \end{pmatrix}_a ,$$

- Action on the reference state:

$$\mathcal{A}_j(u)|0\rangle = \Delta_j(u)|0\rangle \,, \quad \mathcal{C}_j(u)|0\rangle = 0$$

$$\langle 0|\mathcal{A}_j(u) = \langle 0|\Delta_j(u) \,, \quad \langle 0|\mathcal{B}_j(u) = 0$$

for $j = 1, 2, 3$ and where

$$\Delta_1(u) = \cosh\left(u + \frac{\eta}{2}\right)\sinh\left(u - \frac{\eta}{2} + \xi^-\right)\sinh\left(u + \frac{\eta}{2} + \xi^-\right)\left(\sinh(u+\eta)\sinh(u+2\eta)\right)^{2N} ,$$

$$\Delta_2(u) = -\frac{\cosh\left(u + \frac{\eta}{2}\right)\sinh(2u)\sinh\left(u + \frac{3\eta}{2} - \xi^-\right)\sinh\left(u - \frac{\eta}{2} + \xi^-\right)}{\sinh(2(u+\eta))}\left(\sinh(u)\sinh(u+\eta)\right)^{2N} ,$$

$$\Delta_3(u) = \frac{\sinh(2u-\eta)\sinh\left(u + \frac{\eta}{2} - \xi^-\right)\sinh\left(u + \frac{3\eta}{2} - \xi^-\right)}{2\sinh\left(u + \frac{\eta}{2}\right)}\left(\sinh(u)\sinh(u-\eta)\right)^{2N}$$

Bethe ansatz: double-row transfer matrix - Tarasov's construction

- Bethe vectors:

$$|\Psi_m(u_1,\ldots,u_m)\rangle = \mathcal{B}_1(u_1)|\Psi_{m-1}(u_2,\ldots,u_m)\rangle$$
$$-\mathcal{B}_2(u_1)\sum_{i=2}^{m}\Gamma_i^{(m)}(u_1,\ldots,u_m)|\Psi_{m-2}(u_2,\ldots,\hat{u}_i,\ldots,u_m)\rangle$$

$$\langle\Psi_m(u_1,\ldots,u_m)| = \langle\Psi_{m-1}(u_2,\ldots,u_m)|\mathcal{C}_1(u_1)$$
$$-\sum_{i=2}^{m}\tilde{\Gamma}_i^{(m)}(u_1,\ldots,u_m)\langle\Psi_{m-2}(u_2,\ldots,\hat{u}_i,\ldots,u_m)|\mathcal{C}_2(u_1)$$

Bethe ansatz: double-row transfer matrix - Tarasov's construction

- Off-shell equations:

$$
\begin{aligned}
&\tau^{(1,1)}(u)|\Psi_m(u_1,\ldots,u_m)\rangle = \Delta(u,u_1,\ldots,u_m)|\Psi_m(u_1,\ldots,u_m)\rangle \\
&+\sum_{j=1}^{m}\mathcal{G}_1(u,u_j)\frac{\bar{\mathcal{E}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-2\eta)}{\bar{\mathcal{Q}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-\eta)}\prod_{p<j}^{m}\Omega(u_j,u_p)\mathcal{B}_1(u)|\Psi_{m-1}(u_1,\ldots,\hat{u}_j,\ldots,u_m)\rangle \\
&+\sum_{j=1}^{m}\mathcal{G}_2(u,u_j)\frac{\bar{\mathcal{E}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-2\eta)}{\bar{\mathcal{Q}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-\eta)}\prod_{p<j}^{m}\Omega(u_j,u_p)\mathcal{B}_3(u)|\Psi_{m-1}(u_1,\ldots,\hat{u}_j,\ldots,u_m)\rangle \\
&+\sum_{j<k}^{m}\mathcal{H}_{jk}^{(m)}(u,u_1,\ldots,u_m)\mathcal{B}_2(u)|\Psi_{m-2}(u_1,\ldots,\hat{u}_j,\ldots,\hat{u}_k,\ldots,u_m)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\langle\Psi_m(u_1,\ldots,u_m)|\tau^{(1,1)}(u) = \langle\Psi_m(u_1,\ldots,u_m)|\Delta(u,u_1,\ldots,u_m) \\
&+\sum_{j=1}^{m}\langle\Psi_{m-1}(u_1,\ldots,\hat{u}_j,\ldots,u_m)|\mathcal{C}_1(u)\prod_{p<j}^{m}\Omega(u_j,u_p)\frac{\bar{\mathcal{E}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-2\eta)}{\bar{\mathcal{Q}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-\eta)}\mathcal{G}_1(u,u_j) \\
&+\sum_{j=1}^{m}\langle\Psi_{m-1}(u_1,\ldots,\hat{u}_j,\ldots,u_m)|\mathcal{C}_3(u)\prod_{p<j}^{m}\Omega(u_j,u_p)\frac{\bar{\mathcal{E}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-2\eta)}{\bar{\mathcal{Q}}_j(u_j)\bar{\mathcal{Q}}_j(u_j-\eta)}\mathcal{G}_2(u,u_j)\cosh(\eta)^2 \\
&+\sum_{j<k}^{m}\langle\Psi_{m-2}(u_1,\ldots,\hat{u}_j,\ldots,\hat{u}_k,\ldots,u_m)|\mathcal{C}_2(u)\mathcal{H}_{jk}^{(m)}(u,u_1,\ldots,u_m)\cosh(\eta)^2
\end{aligned}
$$

Bethe ansatz: double-row transfer matrix - Tarasov's construction

- Bethe ansatz and Baxter polynomials in this case:

$$
\begin{aligned}
\bar{\mathcal{E}}_j(u) &= \sinh(2u)\sinh\left(u+\frac{\eta}{2}-\xi^+\right)\Delta_1(u)\bar{\mathcal{Q}}_j(u-\eta) \\
&+ \sinh(2(u+\eta))\sinh\left(u+\frac{3\eta}{2}+\xi^+\right)\Delta_2(u)\bar{\mathcal{Q}}_j(u+\eta)
\end{aligned}
$$

$$
\bar{\mathcal{Q}}(u) = \prod_{i=1}^{m} \sinh(u-u_i)\sinh(u+u_i+2\eta)
$$

$$
\bar{\mathcal{Q}}_j(u) = \prod_{i\neq j} \sinh(u-u_i)\sinh(u+u_i+2\eta)
$$

Bethe ansatz: double-row transfer matrix - Tarasov's construction

- The fusion relation

$$U_{\langle 12\rangle}^{(1,1)}(u)$$

$$= \frac{1}{2\sinh(u+\frac{\eta}{2})} F_{\langle 12\rangle}\, U_1^{(\frac{1}{2},1)}(u+\tfrac{\eta}{2})\, R_{12}^{(\frac{1}{2},\frac{1}{2})}(2u)\, U_2^{(\frac{1}{2},1)}(u-\tfrac{\eta}{2})\, \mathcal{P}_{12}\, E_{\langle 12\rangle}$$

allows us to relate $|\Psi_m(u_1,\ldots,u_m)\rangle$ and $|\Phi_m(u_1,\ldots,u_m)\rangle$:

$$|\Psi_m(u_1,\ldots,u_m)\rangle = \bar{s}(u_1,\ldots,u_m)\Big|\Phi_m\left(u_1+\frac{\eta}{2},\ldots,u_m+\frac{\eta}{2}\right)\Big\rangle$$

and

$$\langle\Psi_m(u_1,\ldots,u_m)| = \Big\langle\Phi_m\left(u_1+\frac{\eta}{2},\ldots,u_m+\frac{\eta}{2}\right)\Big|\bar{s}(u_1,\ldots,u_m)\cosh(\eta)^m$$

where

$$\bar{s}(u_1,\ldots,u_m) = 2^{-\frac{m}{2}}\prod_{i=1}^{m}\frac{\sinh(2u_i)\delta_1\left(u_i-\frac{\eta}{2}\right)}{\sinh\left(u_i+\frac{\eta}{2}\right)}\prod_{j<k}^{m}\frac{\sinh(u_j+u_k)\sinh(u_j-u_k-2\eta)}{\sinh(u_j+u_k+\eta)\sinh(u_j-u_k-\eta)}$$

Bethe ansatz: double-row transfer matrix - Tarasov's construction

- The Slavnov scalar product for the double-row Tarasov-Bethe vector is,

$$\langle\Psi_m(u_1,\dots,u_m)|\Psi_m(v_1,\dots,v_m)\rangle = \cosh(\eta)^m \bar{s}(u_1,\dots,u_m)\bar{s}(v_1,\dots,v_m)$$
$$\times\prod_{i=1}^{m}\frac{\delta_2\left(u_i+\frac{\eta}{2}\right)}{\sinh\left(2\left(v_i+\frac{3\eta}{2}\right)\right)\sinh\left(u_i+\frac{\eta}{2}-\xi^+\right)}\prod_{j<i}^{m}\frac{\sinh(u_i+u_j+3\eta)}{\sinh(u_i+u_j+\eta)}$$
$$\times\frac{\det_m\left(\frac{\partial}{\partial u_i}\delta\left(v_j+\frac{\eta}{2},u_1+\frac{\eta}{2},\dots,u_m+\frac{\eta}{2}\right)\right)}{\det_m\left(\bar{\mathcal{F}}(v_i,u_j)\right)}, \qquad (3.2)$$

or

$$\langle\Psi_m(u_1,\dots,u_m)|\Psi_m(v_1,\dots,v_m)\rangle$$
$$=\prod_{j<k}^{m}\frac{\sinh(u_j+u_k)\sinh(u_j-u_k-2\eta)}{\sinh(u_j+u_k+\eta)\sinh(u_j-u_k-\eta)}\frac{\sinh(v_j+v_k)\sinh(v_j-v_k-2\eta)}{\sinh(v_j+v_k+\eta)\sinh(v_j-v_k-\eta)}$$
$$\times\prod_{i=1}^{m}\frac{\sinh(2u_i+2\eta)\Delta_2(u_i)}{\sinh(2v_i+2\eta)\sinh\left(u_i+\frac{\eta}{2}\right)}\prod_{j<i}^{m}\frac{\sinh(u_i+u_j+3\eta)}{\sinh(u_i+u_j+\eta)}$$
$$\times\ \det_m\left(\sinh(2\eta)\bar{\mathcal{F}}(u_i,v_j)\bar{\mathcal{F}}(v_j,u_i)\frac{\bar{\mathcal{E}}_i(v_j)}{\bar{\mathcal{Q}}_i(v_j)}\right)$$

Conclusion

- We have considered the solution of the ZF model, i.e., the obtainment of the Bethe vectors, from two different perspectives. They are simply related, and that allowed us to obtain the Slavnov formulas for Tarasov-Bethe vectors.

- The formulas were written in terms of the Bethe and Baxter polynomials, instead of the derivative of the eigenvalue of the transfer matrix.

- We hope that such kind of formulas also exist for other 19-vertex models, in particular the Izergin-Korepin model. We may expect simplifications in the quantum-group-invariant or root of unity cases.

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Thank you!