Fredholm determinants and long time asymptotics for the nonlinear Luttinger liquid at nonzero temperature

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Luttinger liquid theory

Extremely simple Hamiltonian: harmonic density waves

$$H = \frac{1}{L} \sum_{q \neq 0} u|q| b_q^\dagger b_q$$

Electrons are coherent states of left and right moving bosons

$$\psi^\dagger(x) \sim \exp\left(\sinh\alpha \sum_{q>0} \frac{1}{\sqrt{q}} b_{-q}^\dagger e^{iqx}\right) \exp\left(\cosh\alpha \sum_{q>0} \frac{1}{\sqrt{q}} b_q^\dagger e^{iqx}\right)$$

Interactions into 'Luttinger parameter' $K = e^{2\alpha}$. $K = 1$ for free fermions.

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Simple Hamiltonian, complicated map from quasiparticles to physical particles

Luttinger liquid fails on dynamical correlation functions

e.g. free chiral Green function

$$G(x,t) = \int_0^\infty \frac{\mathrm{d}k}{2\pi} e^{ikx - i\frac{k^2 t}{2m}}$$

Equivalently, Luttinger liquid introduces artificial degeneracy amongst all chiral states at fixed momentum

$$H_R = \frac{1}{L}\sum_{q>0} uqb_q^\dagger b_q = uP.$$

Energy equivalent to momentum $\implies$ basic dynamical correlation functions are delta peaks. $O(k,\omega) = O(k)\delta(\omega - uk)$.

What is the true lineshape?

Imambekov, Glazman, Kamenev, Pustilnik, Affleck, Sirker, Caux, Pereira, Essler, Schneider, . . . , Maillet et al

Hidden Fermi liquid

Recap: (right moving) physical fermions

$$\psi^\dagger_{\mathrm{Phys}}(x) \sim \exp\left(\cosh\alpha \sum_{q>0} \frac{1}{\sqrt{q}} b^\dagger_q e^{iqx}\right)$$

versus quasiparticles

$$\psi^\dagger_{\mathrm{QP}}(x) \sim \exp\left(\sum_{q>0} \frac{1}{\sqrt{q}} b^\dagger_q e^{iqx}\right)$$

with free fermion "nonlinear Luttinger liquid" Hamiltonian

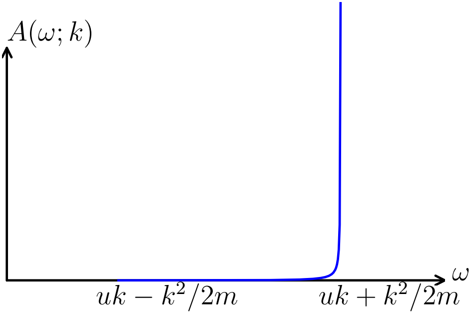
$$H = -\frac{1}{2m}\psi^\dagger_{\mathrm{QP}}\partial_x^2\psi_{\mathrm{QP}}(x) = \frac{1}{3!}\frac{1}{m}(\partial_x\phi)^3.$$

Overlap of electron and quasiparticle $\langle \psi_{\mathrm{QP}}\psi^\dagger_{\mathrm{Phys}}\rangle \sim L^{1-\cosh\alpha} \to 0$

$\to$ "orthogonal (hidden) Fermi liquid"

Singularities in spectral function

Orthogonality determines shape of spectral function at threshold:



$$A(k, \omega) \sim [\omega - \epsilon(k)]^{(\cosh\alpha - 1)^2 - 1}$$

Orthogonality catastrophe softens delta peak into power law.

Imambekov & Glazman, 2009

Summing the Lehmann series to a Fredholm determinant

Lehmann expansion on free fermion states $|\lambda\rangle$:

$$G(x,t) = \sum_{\lambda} e^{-i\epsilon_\lambda t} |\langle\lambda|e^{-i\eta\phi(x)}|0\rangle|^2$$

Wick's theorem: amplitude to create n particle–hole pairs is $\det \langle p_i, q_j|e^{-i\eta\phi(x)}|0\rangle_{i,j=1}^{n}$. This gives the compact expression

$$G(x,t) = \det[1 + L].$$

Identical to Maillet et al series for Lieb–Liniger with shift function fixed to Fermi point $\eta = F(\lambda_F|\lambda_F)$.

Exact solution: Painlevé IV

Fredholm determinant has "integrable" kernel $\Longrightarrow$ a recipe to cook up differential equations for $G(x,t)$.

Its, Izergin, Korepin, Slavnov; Jimbo, Miwa

We find an ODE in scaling variable $s = x/\sqrt{t/m}$

$$\partial_s^2 \log G = \bar{\zeta}\zeta,$$

where $\zeta^{-1}\partial_s\zeta, \bar{\zeta}^{-1}\partial_s\bar{\zeta}$ each solve Painlevé IV

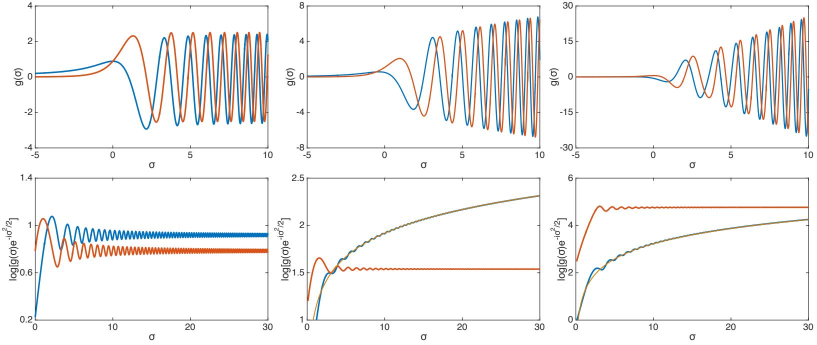
$$\frac{\varphi''}{\varphi} = \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 + \frac{3}{2}\varphi^2 \mp 2is\varphi - \frac{1}{2}s^2 \mp i[\pm 2\eta - 1].$$

The function $i\partial_s \log G$ satisfies the Jimbo–Miwa σ form of Painlevé IV

$$0 = (c'')^2 + (sc' - c)^2 - 4ic'(c' + \eta)^2.$$

Numerical solutions to Painlevé IV

Scaling function $g(x/\sqrt{t/m}) = t^{\eta^2/2}\,\langle \psi_{\mathrm{Phys}}(x,t)\psi^{\dagger}_{\mathrm{Phys}}(0,0)\rangle$



$\eta = 1$ $\qquad$ $\eta = 1.2$ $\qquad$ $\eta = \sqrt{3}$

Finite temperature

Bettelheim, Abanov, and Wiegmann used operator methods to show the thermal correlation functions

$$G_\eta(x, t|\beta) = \mathrm{Tr}\, \rho\, e^{i\eta\phi(x,t)} e^{-i\eta\phi(0,0)}$$

satisfy classical NLSEs

$$\begin{aligned} i\partial_t \Psi + \frac{1}{2m}\partial_x^2 \Psi &= \frac{1}{m}\Psi\bar{\Psi}\Psi \\ -i\partial_t \Psi + \frac{1}{2m}\partial_x^2 \Psi &= \frac{1}{m}\bar{\Psi}\Psi\bar{\Psi} \end{aligned}$$

where $\Psi = G_{\eta+1}/G_\eta$, $\bar{\Psi} = G_{\eta-1}/G_\eta$. The system is closed by the Toda equation

$$\partial_x^2 \log G(x, t) = \bar{\Psi}\Psi.$$

At $T = 0$ these reduce to Painlevé IV.

Finite temperature determinant

For conformal density matrix $\rho = Z^{-1} e^{-\beta H_{LL}}$ we can use the $t = 0$ results for the Green function $G_\eta(x) = \left(\sinh \frac{Tx}{u}\right)^{-\eta^2}$ to "integrate" the NLSEs to Fredholm determinants $G_\eta = \det(1 + L)$,

$$L(k_i, k_j) = \frac{f_r(k_i)\sigma_3^{rs} g_s(k_j)}{k_i - k_j},$$

are determined by the Fermi weight $\vartheta(k) = (1 + e^{\beta vk})^{-1}$,

$$f_1(k) = g_2(k) = \sqrt{1 - \vartheta(k)}\mathcal{F}(k) e^{-\frac{i}{2}\left[\frac{k^2 t}{2m} - kx\right]},$$

$$f_2(k) = g_1(k) = \sqrt{1 - \vartheta(k)}\mathcal{F}(k) e^{-\frac{i}{2}\left[\frac{k^2 t}{2m} - kx\right]} Q(k),$$

Finite temperature determinant

The "form factors" entering also carry temperature dependence

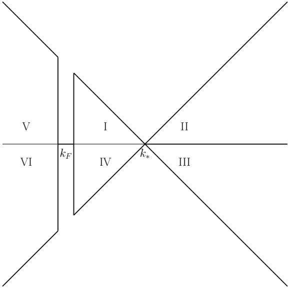
$$\mathcal{F}(k) \equiv \frac{\Gamma\left(\frac{1}{2} + \eta - i\frac{\beta v k}{2\pi}\right)}{\Gamma(\eta)\Gamma\left(\frac{1}{2} - i\frac{\beta v k}{2\pi}\right)}, \; \mathcal{G}(k) \equiv \frac{\Gamma\left(\frac{1}{2} - \eta + i\frac{\beta v k}{2\pi}\right)}{\Gamma(1-\eta)\Gamma\left(\frac{1}{2} + i\frac{\beta v k}{2\pi}\right)}.$$

and Q function

$$Q(z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{z-q} \vartheta(q)\mathcal{G}(q)^2 e^{i\left[\frac{q^2 t}{2m} - qx\right]}.$$

Asymptotics: Riemann–Hilbert problem

Long time behaviour can be extracted by finding an asymptotic solution to the associated Riemann–Hilbert problem
$m^{+}(z) = m^{-}(z)v(z|x,t)$, $m(z) \to 1$ as $z \to \infty$



Solution dominated by poles near the Fermi point that fall within slotted contour, and saddle point $k_* = mx/t$.

Asymptotics: Riemann–Hilbert problem

"Well known" asymptotics of NLSEs come from saddle point

$$\Psi(x,t) \sim \frac{1}{\sqrt{t}} A(k_*) e^{i\frac{k_*^2 t}{2m} - i\nu \log t},$$

The prefactor to the $\log t$ vanishes at zero temperature,

$$\nu = \frac{1}{2\pi} \log\left[1 - (2\pi)^2 \vartheta(k_*)[1-\vartheta(k_*)]\mathcal{F}(k_*)\mathcal{G}(k_*)\right].$$

From the RHP we learn that for $\eta \in (-1/2, 1/2)$, $G_\eta(x,t)$ is determined entirely by poles near the Fermi point. This means

$$G_\eta(x,t) \sim G_\eta(x,0) = \left(\sinh\frac{Tx}{v}\right)^{-\eta^2}, \ \eta \in (-1/2, 1/2).$$

Asymptotics: results

More interesting are the asymptotics for $G_{\eta+1}$,

$$
\begin{aligned}
G_{\eta+1}(x,t) &= G_\eta(x,t)\Psi(x,t) \\
&\sim \left(\sinh\frac{Tx}{v}\right)^{-\eta^2}\frac{1}{\sqrt{t}}A(k_*)e^{i\frac{k_*^2 t}{2m}-i\nu\log t}.
\end{aligned}
$$

At $T=0$ these agree with the Imambekov–Glazman exponents. Explicit form of correction to the finite T mobile impurity phenomenology in the $i\nu\log t$ term. Physical interpretation?

Conclusions

- New (?) determinant representation for thermal correlation functions in nonlinear Luttinger liquid
- New (?) large x, t asymptotics at finite temperatures: explicit form of $e^{-\beta\epsilon(k_*)}$ corrections to thermal mobile impurity phenomenology.

Some open questions:

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- Extension to non conformal density matrix?
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Thank you!