Bethe vectors, scalar products and form factors for integrable models with higher rank algebras

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Generalized integrable models
Notations
Bethe vectors
Scalar products
-Reshetikhin form.
-Det. formula
-Gaudin det.
Form factors
-Twisted scal. prod. trick
-Zero mode method
-Coproduct formula
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General goal

Compute the correlation functions $<\mathcal{O}_1 \cdots \mathcal{O}_n> = tr\Big(\mathcal{O}_1 \cdots \mathcal{O}_n\Big)$ for some local operators $\mathcal{O}_1, \cdots, \mathcal{O}_n$

If one has a basis of the space of states $\mathcal{H}$, $\{|\psi>\}$, then it is enough to compute $<\psi|\mathcal{O}_1 \cdots \mathcal{O}_n|\psi>$ (and then sum on ψ's)

Since we have a basis $\mathcal{O}|\psi>$ can be expressed as a linear combination of ψ's. Thus, to get the correlation function, we need "only":

1. The basis $|\psi>$
2. The decomposition $\mathcal{O}|\psi> = \sum_{\psi'} \mathcal{O}_{\psi\psi'} |\psi'>$
3. The scalar product $<\psi|\psi'>$
4. The form factor $<\psi'|\mathcal{O}|\psi>$

We want to compute the scalar product and the form factors for integrable models associated to algebras with rank > 2

Plan of the talk

- Framework: generalized models
- Bethe vectors (BVs)
- Scalar products of BVs
  - Reshetikhin formula
  - Determinant form
  - Gaudin determinant
- Form Factors (FF)
  - Twisted scalar product tricks
  - Zero mode method
  - Coproduct formula
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Calculations are rather technical $\Rightarrow$ ideas & results only!

General background: generalized quantum integrable models

R-matrix

$R(z_1, z_2) \in V \otimes V$ with $V = End(\mathbb{C}^{\mathfrak{n}})$ and $z_1, z_2 \in \mathbb{C}$ spectral parameters

$R(z_1, z_2)$ obeys Yang-Baxter equation in $V \otimes V \otimes V$

$$R^{12}(z_1, z_2)\, R^{13}(z_1, z_3)\, R^{23}(z_2, z_3) = R^{23}(z_2, z_3)\, R^{13}(z_1, z_3)\, R^{12}(z_1, z_2)$$

$R^{12} = R \otimes \mathbb{I}_{\mathfrak{n}} \;\in\; V \otimes V \otimes V,\; R^{23} = \mathbb{I}_{\mathfrak{n}} \otimes R \;\in\; V \otimes V \otimes V,\, ..$

Universal monodromy matrix $T(x) \in V \otimes \mathcal{A}$

Defines the algebra $\mathcal{A} = Y(gl_{\mathfrak{m}}),\; U_q(\widehat{gl}_{\mathfrak{m}}),\; Y(gl_{\mathfrak{m}|\mathfrak{p}}), \ldots$

$$R^{12}(z_1, z_2)\, T^1(z_1)\, T^2(z_2) = T^2(z_2)\, T^1(z_1)\, R^{12}(z_1, z_2)$$

$$T(z) = \sum_{i,j=1}^{\mathfrak{n}} e_{ij} \otimes T_{ij}(z) \in V \otimes \mathcal{A}[[z^{-1}]]\,,$$

$T^1(z_1) = T(z_1) \otimes \mathbb{I}_{\mathfrak{n}} \;\in\; V \otimes V \otimes \mathcal{A};\; T^2(z_2) = \mathbb{I}_{\mathfrak{n}} \otimes T(z_2) \;\in\; V \otimes V \otimes \mathcal{A}$

$\mathfrak{n} = \text{rank}\mathcal{A}$ (i.e. $\mathfrak{n} = \mathfrak{m}, \mathfrak{m} + \mathfrak{p}, \ldots$)

Generalized quantum integrable models

Transfer matrix $\mathfrak{t}(z)$

For algebras

$$\mathfrak{t}(z) = \text{Tr } T(z) = T_{11}(z) + ... + T_{\mathfrak{nn}}(z)$$

For superalgebras

$$\begin{aligned}\mathfrak{t}(z) &= \text{sTr } T(z) = \sum_{i=1}^{\mathfrak{n}} (-1)^{[i]} T_{ii}(z) \\ &= T_{11}(z) + ... + T_{\mathfrak{mm}}(z) - T_{\mathfrak{m}+1,\mathfrak{m}+1}(z) - ... - T_{\mathfrak{p}+\mathfrak{m},\mathfrak{p}+\mathfrak{m}}(z)\end{aligned}$$

Due to YBE $[\mathfrak{t}(z)\,,\,\mathfrak{t}(z')] = 0$, the transfer matrix defines an integrable model (with periodic boundary conditions).

In general, the Hamiltonian is chosen as $H = -\frac{d}{dz} \ln \mathfrak{t}(z)\Big|_{z=z_0}$.

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Choice of a (lowest weight) representation $\mathcal{A}$:

$$T_{jj}(z)|0\rangle = \lambda_j(z)|0\rangle,\ j = 1,..,\mathfrak{n} \qquad T_{ij}(z)|0\rangle = 0, \qquad 1 \leq j < i \leq \mathfrak{n}$$

Up to normalisation of $T(z)$, we only need the ratios

$$r_i(z) = \frac{\lambda_i(z)}{\lambda_{i+1}(z)}, \qquad i = 1, ..., \mathfrak{n} - 1.$$

we keep $r_i(z)$ as free functional parameters.

The calculation is valid for any representation (provided it is lowest/highest weight): these are the generalized models

Example: the "fundamental" spin chain

For $\mathcal{A} = Y(gl_{\mathfrak{m}})$, $U_q(\widehat{gl}_{\mathfrak{m}})$, $Y(gl_{\mathfrak{m}|\mathfrak{p}})$, ..., we consider the following monodromy matrix:

$$\begin{aligned} T^{0}(z|\bar{z}) &= R^{01}(z-z_1)R^{02}(z-z_2)\cdots R^{0L}(z-z_L) \\ \lambda_1(z) &= \prod_{\ell=1}^{L}\left(1-\frac{1}{z-z_\ell}\right) \\ \lambda_j(z) &= 1 \qquad j=2,...,\mathfrak{n} \end{aligned}$$

It corresponds to a periodic spin chain with L sites, each of them carrying a fundamental representation of $\mathcal{A}$.

- $1, 2, ..., L$ are the quantum (physical) spaces of the spin chain. Here they are $\mathfrak{n}$-dimensional: on each site the "spins" can take $\mathfrak{n}$ values.
- $\bar{z} = \{z_1, ..., z_L\}$ are the inhomogeneities.
- 0 is the auxiliary space.

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To illustrate the talk, we will focus on $Y(gl_{\mathfrak{n}})$ (rational R-matrix) and possibly take $\mathfrak{n} = 3$:

$$
\begin{aligned}
R(z_1, z_2) &= \mathbf{I} + g(z_1, z_2)\,\mathbf{P} \in End(\mathbb{C}^3) \otimes End(\mathbb{C}^3) \\
g(z_1, z_2) &= \frac{c}{z_1 - z_2}
\end{aligned}
$$

$\mathbf{I}$ is the identity matrix, $\mathbf{P}$ is the permutation matrix between two spaces $End(\mathbb{C}^3)$, c is a constant.
It corresponds to XXX-like models and is based on $Y(gl_3)$.

$$
R(z_1, z_2) = \left(\begin{array}{ccc|ccc|ccc}
f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & g & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & g & 0 & 0 \\
\hline
0 & g & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & g & 0 \\
\hline
0 & 0 & g & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & g & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f
\end{array}\right)
$$

$Y(gl_3)$: $g \equiv g(z_1, z_2)$ and $f \equiv f(z_1, z_2) = 1 + g(z_1, z_2)$

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Notations

We have already introduced

- The functions

$$g(z_1, z_2) = \frac{c}{z_1 - z_2} \quad \text{and} \quad f(z_1, z_2) = \frac{z_1 - z_2 + c}{z_1 - z_2},$$

that enter in the definition of the R-matrix
  ⇒ The interaction in the bulk (XXX or XXZ type).

- The free functionals

$$r_i(z) = \frac{\lambda_i(z)}{\lambda_{i+1}(z)}, \qquad i = 1, ..., \mathfrak{n} - 1$$

that (potentially) describe the representation
  ⇒ The type of spin chain (spins and length of the chain).

We also use

$$h(z_1, z_2) = \frac{f(z_1, z_2)}{g(z_1, z_2)}, \quad t(z_1, z_2) = \frac{g(z_1, z_2)}{h(z_1, z_2)}.$$

Notations

Many sets of variables (☹ ... don't be scared)

- "bar" always denote sets of variables: $\bar{w}$, $\bar{u}$, $\bar{v}$ etc..
- Individual elements of the sets have latin subscripts: w_j, u_k, etc..
- # is the cardinality of a set: $\bar{w} = \{w_1, w_2\} \;\Rightarrow\; \#\bar{w} = 2$, etc...
- Subsets of variables are denoted by roman indices: $\bar{u}_{\mathrm{I}}$, $\bar{v}_{\mathrm{iv}}$, $\bar{w}_{\mathrm{II}}$, etc.
- Special case: $\bar{u}_j = \bar{u} \setminus \{u_j\}$, $\bar{w}_k = \bar{w} \setminus \{w_k\}$, etc...

Notations

Many sets of variables ☺

- "bar" always denote sets of variables: $\bar{w}$, $\bar{u}$, $\bar{v}$ etc..
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- $\#$ is the cardinality of a set: $\bar{w} = \{w_1, w_2\} \Rightarrow \#\bar{w} = 2$, etc...
- Subsets of variables are denoted by roman indices: $\bar{u}_{\rm I}$, $\bar{v}_{\rm iv}$, $\bar{w}_{\rm II}$, etc.
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Shorthand notations for products of scalar functions (when they depend on one or two variables):

$$f(\bar{u}_{\rm II}, \bar{u}_{\rm I}) = \prod_{u_j \in \bar{u}_{\rm II}} \prod_{u_k \in \bar{u}_{\rm I}} f(u_j, u_k),$$

$$r_1(\bar{u}_{\rm II}) = \prod_{u_j \in \bar{u}_{\rm II}} r_1(u_j); \quad g(v_k, \bar{w}) = \prod_{w_j \in \bar{w}} g(v_k, w_j), \quad etc..$$

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Bethe vectors (BVs)

Framework: Algebraic-Nested Bethe Ansatz (Leningrad school 80's)
[Faddeev, Kulish, Reshetikhin, Sklyanin, Takhtajan, ...]

For "usual" XXX (gl_2) spin chain

Only one 'raising' operator $T_{12}(z)$ and one set of Bethe parameters $\bar{u}$:

$$\mathbb{B}^a(\bar{u}) = T_{12}(u_1)T_{12}(u_2)\cdots T_{12}(u_a)|0\rangle,$$

For higher rank $\mathfrak{n}$:

There are many raising operators $T_{i,i+1}(z)$, $i = 1, 2, ..., \mathfrak{n}-1$.
There are $\mathfrak{n}-1$ sets of Bethe parameters:

$$\bar{t}^{(j)} = \{t_1^{(j)}, ..., t_{a_j}^{(j)}\}, \quad \#\bar{t}^{(j)} = a_j \in \mathbb{Z}_+, \quad j = 1, 2, ..., \mathfrak{n}-1$$

$$\bar{t} = \{\bar{t}^{(1)}, \bar{t}^{(2)}, ..., \bar{t}^{(\mathfrak{n}-1)}\}, \quad \bar{a} = \{a_1, a_2, ..., a_{\mathfrak{n}-1}\}$$

$\mathbb{B}^{\bar{a}}(\bar{t})$ appears to be much more complicated...

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On-shell Bethe vectors and Bethe equations

$\mathbb{B}^{\bar{a}}(\bar{t})$ is a transfer matrix eigenvector

$$\mathfrak{t}(z)\,\mathbb{B}^{\bar{a}}(\bar{t}) = \tau(z|\bar{t})\,\mathbb{B}^{\bar{a}}(\bar{t})$$

provided the Bethe equations (BAEs) are obeyed:

$$r_i(\bar{t}^{(i)}_{\mathrm{I}}) = \frac{f(\bar{t}^{(i)}_{\mathrm{I}},\bar{t}^{(i)}_{\mathrm{II}})}{f(\bar{t}^{(i)}_{\mathrm{II}},\bar{t}^{(i)}_{\mathrm{I}})}\,\frac{f(\bar{t}^{(i+1)},\bar{t}^{(i)}_{\mathrm{I}})}{f(\bar{t}^{(i)}_{\mathrm{I}},\bar{t}^{(i-1)})}, \quad i=1,...,\mathfrak{n}-1$$

$$\text{with} \quad \bar{t}^{(0)} = \emptyset = \bar{t}^{(\mathfrak{n})}$$

that hold for arbitrary partitions of the sets $\bar{t}^{(j)}$ into subsets $\{\bar{t}^{(j)}_{\mathrm{I}},\ \bar{t}^{(j)}_{\mathrm{II}}\}$.

In that case, $\mathbb{B}^{\bar{a}}(\bar{t})$ will be called an on-shell BV

Generalized models

The Bethe equations are not seen as a 'quantization' of the Bethe parameters $\bar{t}$ anymore
But rather as functional relations on the functions $r_i(x)$, $i=1,...,\mathfrak{n}-1$.

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Presentation for $Y(gl_3)$: $\bar{t}^{(1)} \equiv \bar{u}$ and $\bar{t}^{(2)} \equiv \bar{v}$

Known formulas: Trace formula ['07 Tarasov & Varchenko]

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \underbrace{tr\Big(\overbrace{\mathbb{T}(\bar{u};\bar{v})\,\mathbb{R}(\bar{u};\bar{v})}^{\in Y(gl_3)\otimes V^{\otimes(a+b)}}\; e_{21}^{\otimes a} \otimes e_{32}^{\otimes b}\Big)}_{\in Y(gl_3)}\,|0\rangle$$

$$\#\bar{u} = a\,, \quad \#\bar{v} = b\,, \quad e_{ij} = 3\times 3 \ \text{ elementary matrices}$$

The trace is taken over $a+b$ auxiliary spaces, i.e. in $End(\mathbb{C}^3)^{\otimes(a+b)}$

$$\begin{aligned}\mathbb{T}(\bar{u},\bar{v}) &= T^1(u_1)\cdots T^a(u_a)\,T^{a+1}(v_1)\cdots T^{a+b}(v_b)\\ \mathbb{R}(\bar{u},\bar{v}) &= \Big(R^{a,a+1}(u_a,v_1)\cdots R^{a,a+b}(u_a,v_b)\Big)\cdots\times\\ &\quad\times\cdots\Big(R^{1,a+1}(u_1,v_1)\cdots R^{1,a+b}(u_1,v_b)\Big)\end{aligned}$$

- Valid for $Y(gl_{\mathfrak{m}})$ and $U_q(\widehat{gl}_{\mathfrak{m}})$.
- Exists also for superalgebras $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{m}|\mathfrak{p}})$

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Recursion formulas

$$\lambda_2(u_k)f(\bar v,u_k)\mathbb{B}^{a+1,b}(\bar u;\bar v) = T_{12}(u_k)\,\mathbb{B}^{a,b}(\bar u_k;\bar v) + + \sum_{i=1}^{b} g(v_i,u_k)f(\bar v_i,v_i)\,T_{13}(u_k)\,\mathbb{B}^{a,b-1}(\bar u_k;\bar v_i),$$

$$\lambda_2(v_k)f(v_k,\bar u)\mathbb{B}^{a,b+1}(\bar u;\bar v) = T_{23}(v_k)\,\mathbb{B}^{a,b}(\bar u;\bar v_k) + + \sum_{j=1}^{a} g(v_k,u_j)f(u_j,\bar u_j)\,T_{13}(v_k)\,\mathbb{B}^{a-1,b}(\bar u_j;\bar v_k).$$

N.B. Since $T_{12}(z)$, $T_{23}(z)$ and $T_{13}(z)$ are associated resp. to the gl_3 roots α, β and $\alpha+\beta$, $\mathbb{B}^{a,b}(\bar u;\bar v)$ "behaves" as the root $a\,\alpha + b\,\beta$.
Cf. gl_2 case: $\mathbb{B}^{a+1}(\bar u) = T_{12}(u_k)\,\mathbb{B}^{a}(\bar u_k)$.

In fact $T_{ij}(z)$ and BVs are eigenvectors of the zero modes $T_{kk}[0]$, see later.

- Valid for $Y(gl_3)$ and $U_q(\widehat{gl}_3)$.
- Exists also for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{n}})$.

Explicit formulas

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \sum \frac{\mathrm{K}_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})}{\lambda_2(\bar{v}_{\rm II})\lambda_2(\bar{u})} \frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})f(\bar{u}_{\rm II},\bar{u}_{\rm I})}{f(\bar{v}_{\rm II},\bar{u})f(\bar{v}_{\rm I},\bar{u}_{\rm I})} T_{12}(\bar{u}_{\rm II})T_{13}(\bar{u}_{\rm I})T_{23}(\bar{v}_{\rm II})|0\rangle$$

(Plus other formulas with different order of T_{12}, T_{13}, T_{23})

The sums are taken over partitions of the sets:
$\bar{u} \Rightarrow \{\bar{u}_{\rm I},\bar{u}_{\rm II}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\rm I},\ \bar{v}_{\rm II}\}$ with $0 \le \#\bar{u}_{\rm I} = \#\bar{v}_{\rm I} = k \le \min(a,b)$.

$\mathrm{K}_k(\bar{v}_{\rm I}|\bar{u}_{\rm I})$ is the **Izergin–Korepin determinant**

$$\mathrm{K}_k(\bar{x}|\bar{y}) = \Delta_k(\bar{x})\,\Delta'_k(\bar{y})\,h(\bar{x},\bar{y})\ \det_k\left[t(x_i,y_j)\right]$$

$$\Delta_k(\bar{x}) = \prod_{\ell<m}^{k} g(x_\ell,x_m) \quad ; \quad \Delta'_k(\bar{y}) = \prod_{\ell<m}^{k} g(y_m,y_\ell)$$

- Valid for $Y(gl_3)$ and $U_q(\widehat{gl}_3)$.
- Exists also for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{n}})$.

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Current presentation and projection method

We use of projectors method in the current realization of $DY(gl_3)$, [Khoroshkin, Pakuliak and collaborators 2006-10]

N.B. The current realization is related to a Gauss decomposition of the monodromy matrix $T(z)$

Explicit expression of BVs in a different basis

$$\mathbb{B}(\bar{u},\bar{v}) = \mathcal{P}_f^+\Big(F_1(u_1)\cdots F_1(u_a)F_2(v_1)\cdots F_2(v_b)\Big)\, K_1(\bar{u})\, K_2(\bar{v})\, |0\rangle$$

- $K_1(z)$ and $K_2(z)$ are the Cartan generators
- $F_1(z)$ generator associated to the first simple (negative) root
- $F_2(z)$ generator associated to the second simple (negative) root
- $\mathcal{P}_f^+$ projector of the Borel subalgebra on the positive modes

Useful to get explicit expressions and recursion relations

- Valid for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{m}})$.

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All these formulas are related

- Explicit expressions obey the recursion formulas
- Trace formula obeys the recursion formulas
- Recursion formulas uniquely fix the BVs, once $\mathbb{B}^{a,0}(\bar{u},.)$ or $\mathbb{B}^{0,b}(.,\bar{v})$ are known.
- The projection of currents coincides with the trace formula

Bethe vectors $\mathbb{B}(\bar{t})$

- On-shell BVs: the BAEs are obeyed so that

$$\mathfrak{t}(z)\,\mathbb{B}(\bar{t}) = \tau(z|\bar{t})\,\mathbb{B}(\bar{t})$$

- Off-shell BVs otherwise

Dual Bethe vectors $\mathbb{C}(\bar{t})$

- On-shell dual BVs: the (same) BAEs are obeyed so that

$$\mathbb{C}(\bar{t})\,\mathfrak{t}(z) = \tau(z|\bar{t})\,\mathbb{C}(\bar{t})$$

- Off-shell dual BVs otherwise

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Scalar products of BVs: Reshetikhin formula

$$S(\bar{s}|\bar{t}) = \mathbb{C}^{\bar{b}}(\bar{s})\,\mathbb{B}^{\bar{a}}(\bar{t})$$

$$\bar{s} = \{\bar{s}^{(1)}, \bar{s}^{(2)}, ...\bar{s}^{(\mathfrak{n}-1)}\}, \quad \#\bar{s}^{(j)} = b_j, \quad \bar{b} = \{b_1, b_2, ..., b_{\mathfrak{n}-1}\}$$
$$\bar{t} = \{\bar{t}^{(1)}, \bar{t}^{(2)}, ...\bar{t}^{(\mathfrak{n}-1)}\}, \quad \#\bar{t}^{(j)} = a_j, \quad \bar{a} = \{a_1, a_2, ..., a_{\mathfrak{n}-1}\}$$

General formula given by Reshetikhin formula

$$S(\bar{s}|\bar{t}) = \sum W_{\rm part}(\bar{s}_{\rm I}, \bar{s}_{\rm II}|\bar{t}_{\rm I}, \bar{t}_{\rm II}) \prod_{j=1}^{\mathfrak{n}-1} r_j(\bar{s}_{\rm I}^{(j)}) r_j(\bar{t}_{\rm II}^{(j)}).$$

The sum is taken over all possible partitions such that $\#\bar{t}_{\rm I}^{(j)} = \#\bar{s}_{\rm I}^{(j)}$. The expression is valid for all BVs (on-shell or off-shell).

- Valid for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{m}})$.

But difficult to handle $\Rightarrow$ we look for determinant expressions for $S(\bar{s}|\bar{t})$.

Scalar products of BVs: determinant formula

Here we consider the case $\mathfrak{n}=3$ and the scalar product of an on-shell Bethe vector

$$\mathfrak{t}(z)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)=\tau(z|\bar{u}^B,\bar{v}^B)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\quad\text{and BAEs}$$

with a *twisted* dual on-shell Bethe vector

$$\mathbb{C}^{a,b}_{\kappa}(\bar{u}^C;\bar{v}^C)\,\mathfrak{t}_{\kappa}(z)=\tau_{\kappa}(z|\bar{u}^C,\bar{v}^C)\,\mathbb{C}^{a,b}_{\kappa}(\bar{u}^C;\bar{v}^C)$$

with twisted BAEs

$$\mathfrak{t}_{\kappa}(z)=tr\big(M\,T(z)\big)\quad\text{with}\quad M=\begin{pmatrix}1&0&0\\0&\kappa&0\\0&0&1\end{pmatrix}$$

$$\mathfrak{t}_{\kappa}(z)=T_{11}(z)+\kappa\,T_{22}(z)+T_{33}(z)$$

$$\mathcal{S}^{a,b}_{\kappa}\equiv\mathbb{C}^{a,b}_{\kappa}(\bar{u}^C;\bar{v}^C)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$$

$$\mathcal{S}_{\kappa}^{a,b} = f(\bar{v}^C, \bar{u}^C) f(\bar{v}^B, \bar{u}^B) t(\bar{v}^C, \bar{u}^B) \Delta'_a(\bar{u}^C) \Delta_a(\bar{u}^B) \Delta'_b(\bar{v}^C) \Delta_b(\bar{v}^B) \times \det_{a+b} \mathcal{M},$$

$$\Delta'_n(\bar{w}) = \prod_{j>k}^{n} g(w_j, w_k), \qquad \Delta_n(\bar{w}) = \prod_{j<k}^{n} g(w_j, w_k).$$

$\mathcal{M}$ is a $(a+b) \times (a+b)$ matrix. For $\bar{\xi} = \{\bar{u}^B, \bar{v}^C\}$:

$$\mathcal{M}_{j,k} = \frac{c}{g(\xi_k, \bar{u}^C) g(\bar{v}^C, \xi_k)} \frac{\partial \tau_{\kappa}(\xi_k | \bar{u}^C, \bar{v}^C)}{\partial u_j^C}, \quad j = 1, \dots, a,$$

$$\mathcal{M}_{a+j,k} = \frac{-c}{g(\xi_k, \bar{u}^B) g(\bar{v}^B, \xi_k)} \frac{\partial \tau(\xi_k | \bar{u}^B, \bar{v}^B)}{\partial v_j^B}, \quad j = 1, \dots, b.$$

- Valid for a twist $\{\kappa_1, \kappa_2, \kappa_3\}$ up to corrections in $(\kappa_i - 1)(\kappa_j - 1)$.
- Exists for $U_q(\widehat{gl}_3)$ for the twist $\{1, \kappa, 1\}$.
- Exists for $Y(gl_{2|1})$ and $Y(gl_{1|2})$.

Norm of on-shell BVs: Gaudin determinant

We present the example $Y(gl_{\mathfrak{n}})$

Rewriting of the BAEs

$$\Phi_k^{(i)} = r_i(t_k^{(i)}) \frac{f(\bar{t}_k^{(i)}, t_k^{(i)})}{f(t_k^{(i)}, \bar{t}_k^{(i)})} \frac{f(t_k^{(i)}, \bar{t}^{(i-1)})}{f(\bar{t}^{(i+1)}, t_k^{(i)})}, \qquad k = 1, \dots, a_i, \quad i = 1, \dots, \mathfrak{n}-1$$

$$\text{BAEs} \quad : \quad \Phi_k^{(i)} = 1$$

The Gaudin matrix

The Gaudin matrix G is a block matrix $\left(G^{(i,j)}\right)_{i,j=1,..,\mathfrak{n}-1}$

Each block $G^{(i,j)}$, of size $a_i \times a_j$, has entries

$$G_{k,l}^{(i,j)} = -c \, \frac{\partial \ln(\Phi_k^{(i)})}{\partial t_l^{(j)}}$$

Norm of $\mathbb{B}(\bar{t})$: $S(\bar{t}) = \mathbb{C}(\bar{t})\mathbb{B}(\bar{t})$

For an on-shell $\mathbb{B}(\bar{t})$

$$S(\bar{t}) \quad = \quad \prod_{i=1}^{\mathfrak{n}} \prod_{k=1}^{a_i} \left(\frac{f(\bar{t}_k^{(i)}, t_k^{(i)})}{f(\bar{t}^{(i+1)}, t_k^{(i)})} \right) \det G$$

N.B. $\mathbb{B}$ and $\mathbb{C}$ have to be on-shell. In that case $\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \delta_{\bar{s},\bar{t}}\, S(\bar{t})$

- Valid for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{m}})$.

Form Factors (FF)

Form factors $\mathcal{F}_{ij}(z|\bar{s};\bar{t})$

$$\mathcal{F}_{ij}(z|\bar{s};\bar{t}) = \mathbb{C}(\bar{s})\,T_{ij}(z)\mathbb{B}(\bar{t}), \qquad i,j = 1,\dots,\mathfrak{n}-1$$

where both $\mathbb{C}(\bar{s})$ and $\mathbb{B}(\bar{t})$ are on-shell Bethe vectors

Three tricks to compute them

- Twisted scalar product trick (diagonal FF)
- Zero mode method (off-diagonal FF)
- Coproduct formula (other FF / Composite models)
- Universal form factor

Again we take $Y(gl_{\mathfrak{n}})$ to give simple formulas.

Bethe vectors, scalar products and form factors for integrable models with higher rank algebras

Eric Ragoucy

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Diagonal form factors: the twisted scalar product trick

Diagonal FF $\mathcal{F}_{jj}(z|\bar{s};\bar{t})$ are computed using the "twisted scalar product" trick

$$t_{\bar{\kappa}}(z) - t(z) = (\kappa_1 - 1)\, T_{11}(z) + \cdots + (\kappa_{\mathfrak{n}} - 1)\, T_{\mathfrak{n}\mathfrak{n}}(z)$$

$$T_{jj}(z) = \frac{d}{d\kappa_j}\Big(t_{\bar{\kappa}}(z) - t(z)\Big)\,, \quad j = 1, 2, ..., \mathfrak{n}$$

$$\mathcal{F}_{jj}(z|\bar{s};\bar{t}) = \frac{d}{d\kappa_j}\Big[\mathbb{C}_{\bar{\kappa}}(\bar{s})\big(t_{\bar{\kappa}}(z) - t(z)\big)\mathbb{B}(\bar{t})\Big]_{\bar{\kappa}=1} = \frac{d}{d\kappa_j}\Big[\big(\tau_{\bar{\kappa}}(z;\bar{s}) - \tau(z;\bar{t})\big)\, \mathcal{S}_{\bar{\kappa}}(\bar{s}|\bar{t})\Big]_{\bar{\kappa}=1}$$

- Valid also for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{m}})$
- Det. form only when $\mathcal{S}_{\bar{\kappa}}(\bar{s}|\bar{t})$ has one: $Y(gl_3)$, $Y(gl_{2|1})$, $U_q(\widehat{gl}_3)$

Form factors (off-diagonal): Zero mode method

Zero modes of the monodromy matrix

$$T_{ij}[0] = \lim_{w \to \infty} \frac{w}{c} T_{ij}(w)$$

They form a $gl_{\mathfrak{n}}$ Lie subalgebra in $Y(gl_{\mathfrak{n}})$

$$\begin{aligned}
\big[T_{ij}[0]\,,\; T_{kl}[0]\big] &= \delta_{kj}\, T_{il}[0] - \delta_{il}\, T_{kj}[0] \\
\big[T_{ij}[0]\,,\; T_{kl}(z)\big] &= \delta_{kj}\, T_{il}(z) - \delta_{il}\, T_{kj}(z)
\end{aligned}$$

Bethe vectors and zero modes

$$\lim_{w \to \infty} \frac{w}{c}\, \mathbb{B}(\bar t^{(1)}, .., \{\bar t^{(j-1)}, w\}, \bar t^{(j)}, ..\bar t^{(\mathfrak{n}-1)}) = T_{j-1,j}[0]\, \mathbb{B}(\bar t),$$

$$\lim_{w \to \infty} w\, \mathbb{C}(\bar s^{(1)}, .., \{\bar s^{(j-1)}, w\}, \bar s^{(j)}, ..\bar s^{(\mathfrak{n}-1)}) = \mathbb{C}(\bar s)\, T_{j,j-1}[0].$$

- Valid also for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{m}})$

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Infinite Bethe roots in $Y(gl_{\mathfrak{m}|\mathfrak{p}})$

The BAEs are compatible with the limit $t_k^{(j)} \to \infty$ for j and k fixed.

$$\Rightarrow \text{If } \mathbb{B}(\{w, \bar{t}\}) \text{ is on-shell then so is } \mathbb{B}(\{\infty, \bar{t}\})$$

Highest weight property of on-shell BVs in $Y(gl_{\mathfrak{m}|\mathfrak{p}})$

If $\mathbb{B}(\bar{t})$ and $\mathbb{C}(\bar{s})$ are on-shell, with $\bar{t}$ and $\bar{s}$ finite, then

$$T_{j,j-1}[0]\,\mathbb{B}(\bar{t}) = 0 \quad \text{and} \quad \mathbb{C}(\bar{s})\,T_{j,j-1}[0] = 0$$

Zero mode method

Idea: *use the Lie algebra symmetry generated by the zero modes and the highest weight property of (on-shell) Bethe vectors to obtain relations among form factors*

$\mathcal{F}_{ij}(z|\bar{s};\bar{t}) = \mathbb{C}(\bar{s})\, T_{ij}(z)\mathbb{B}(\bar{t})$

$$
\begin{aligned}
&\lim_{w\to\infty} \frac{w}{c}\mathcal{F}_{jj}(z|\bar{s};\bar{t}^{(1)},..,\{\bar{t}^{(j-1)},w\},\bar{t}^{(j)},..\bar{t}^{(n-1)}) \\
&= \mathbb{C}(\bar{s})\, T_{jj}(z) \lim_{w\to\infty}\frac{w}{c}\,\mathbb{B}(\bar{t}^{(1)},..,\{\bar{t}^{(j-1)},w\},\bar{t}^{(j)},..\bar{t}^{(n-1)})) \\
&= \mathbb{C}(\bar{s})\, T_{jj}(z)\, T_{j-1,j}[0]\,\mathbb{B}(\bar{t}) \\
&= \mathbb{C}(\bar{s})\, \big[T_{jj}(z)\,,\; T_{j-1,j}[0]\big]\,\mathbb{B}(\bar{t}) \\
&= \mathbb{C}(\bar{s})\, T_{j-1,j}(z)\,\mathbb{B}(\bar{t})) \\
&= \mathcal{F}_{j-1,j}(z|\bar{s};\bar{t}).
\end{aligned}
$$

Symbolically: $\lim_{w\to\infty}\frac{w}{c}\mathcal{F}_{jj}(z|\bar{s};\{w,\bar{t}\}) = \mathcal{F}_{j-1,j}(z|\bar{s};\bar{t}), \quad w\in\bar{t}^{(j)}$

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Form factors (off-diagonal case)

$$\mathcal{F}_{i,j}(z|\bar{s},\bar{t}) = \mathbb{C}(\bar{s})T_{ij}(z)\mathbb{B}(\bar{t}), \qquad i \neq j$$

$$\lim_{w\to\infty} \frac{w}{c}\mathcal{F}_{jj}(z|\bar{s};\{w,\bar{t}\}) = \mathcal{F}_{j-1,j}(z|\bar{s};\bar{t}), \qquad w \in \bar{t}^{(j)}$$

$$\lim_{w\to\infty} \frac{w}{c}\mathcal{F}_{jj}(z|\{w,\bar{s}\};\bar{t}) = -\mathcal{F}_{j,j-1}(z|\bar{s};\bar{t}), \qquad w \in \bar{s}^{(j)}$$

$$\lim_{w\to\infty} \frac{w}{c}\mathcal{F}_{j-1,j}(z|\bar{s};\{w,\bar{t}\}) = \mathcal{F}_{j-2,j}(z|\bar{s};\bar{t}), \qquad w \in \bar{t}^{(j-1)}$$

$$\lim_{w\to\infty} \frac{w}{c}\mathcal{F}_{j,j-1}(z|\{w,\bar{s}\};\bar{t}) = -\mathcal{F}_{j,j-2}(z|\bar{s};\bar{t}), \qquad w \in \bar{s}^{(j-1)}$$

etc...

$\Rightarrow$ All off-diagonal FF can be deduced from diagonal ones

$$\lim_{w\to\infty} \frac{w}{c}\mathcal{F}_{j-1,j}(z|\{w,\bar{s}\};\bar{t}) = \mathcal{F}_{j,j}(z|\bar{s};\bar{t}) - \mathcal{F}_{j-1,j-1}(z|\bar{s};\bar{t}), \quad w \in \bar{s}^{(j)}$$

$\Rightarrow$ Altogether only one diagonal FF is needed!

- Valid for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$.
- May be adapted to $U_q(\widehat{gl}_{\mathfrak{m}})$...

Coproduct formula and composite models

Local Form Factors

$$T(z) = T^{(2)}(z)\, T^{(1)}(z) \quad \text{with} \quad \begin{array}{l} T^{(2)}(z) = \mathcal{L}_L(z) \cdots \mathcal{L}_{m+1}(z) \\ T^{(1)}(z) = \mathcal{L}_m(z) \cdots \mathcal{L}_1(z) \end{array}$$

$m \in [1, L[$ plays the role of a position x in a continuous version.
$T^{(j)}(z)$ are monodromy matrices for "shorter chains".

Coproduct formula

$$\mathbb{B}(\bar{u}; \bar{v}) = \sum \frac{\ell_3(\bar{v}_{\rm II})}{\ell_1(\bar{u}_{\rm I})} f(\bar{u}_{\rm I}, \bar{u}_{\rm II}) f(\bar{v}_{\rm II}, \bar{v}_{\rm I}) f(\bar{v}_{\rm I}, \bar{u}_{\rm I}) \times \mathbb{B}^{(1)}(\bar{u}_{\rm I}; \bar{v}_{\rm I})\, \mathbb{B}^{(2)}(\bar{u}_{\rm II}; \bar{v}_{\rm II}),$$

where $\mathbb{B}^{(j)}(\bar{u}; \bar{v})$ are Bethe vectors for $T^{(j)}(z)$, $j = 1, 2$.

N.B. $\mathbb{B}^{(1)}(\bar{u}; \bar{v})$ and $\mathbb{B}^{(2)}(\bar{u}; \bar{v})$ are off-shell even when $\mathbb{B}(\bar{u}; \bar{v})$ is on-shell.

- Valid for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ and $U_q(\widehat{gl}_{\mathfrak{m}})$.

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Universal Form Factors

We consider the FF $\mathcal{F}_{i,j}(z|\bar{s},\bar{t}) = \mathbb{C}(\bar{s})\,T_{ij}(z)\mathbb{B}(\bar{t})$

When $\mathbb{C}(\bar{s})$ and $\mathbb{B}(\bar{t})$ are on-shell and such that their eigenvalues $\tau(z|\bar{s})$ and $\tau(z|\bar{t})$ are different

$$\mathbb{F}_{i,j}(\bar{s},\bar{t}) = \frac{\mathcal{F}_{i,j}(z|\bar{s},\bar{t})}{\tau(z|\bar{s}) - \tau(z|\bar{t})}$$

is independent of z and does not depend on the monodromy matrix. It depends solely on the R-matrix $\Rightarrow$ It is model independent.

- Valid for $Y(gl_{\mathfrak{m}|\mathfrak{p}})$.

Summary (1/3): Bethe Vectors

- Expressions for (off-shell) Bethe vectors and their duals
  - Trace formula: $Y(gl_{\mathfrak{m}|\mathfrak{n}})$ and $U_q(\widehat{gl}_{\mathfrak{m}|\mathfrak{n}})$ [TV 07 / BR 08]
  - Bethe vectors as projections on Borel subalgebras $U_q(\widehat{gl}_{\mathfrak{n}})$ [KP 06-10] and $Y(gl_{\mathfrak{m}|\mathfrak{n}})$ [1611.09620]
  - Explicit expressions:
    - $Y(gl_{\mathbf{3}})$ [1210.0768], $U_q(\widehat{gl}_{\mathbf{3}})$ [1012.1455], $Y(gl_{\mathbf{2|1}})$ [1604.02311]
    - $Y(gl_{\mathfrak{m}|\mathfrak{n}})$ [1611.09620] and $U_q(\widehat{gl}_{\mathfrak{n}})$ [1310.3253]
- Action of $T_{ij}(\bar{x})$ on BVs (and/or recursion relations)
  - $Y(gl_3)$ [1210.0768], $U_q(\widehat{gl}_3)$ [1210.0768],
  - $Y(gl_{2|1})$ [1605.06419]
  - $Y(gl_{\mathfrak{m}|\mathfrak{p}})$ [1611.09620] and $U_q(\widehat{gl}_{\mathfrak{m}})$ [1310.3253] from proj. method

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Summary (2/3): Scalar Products

- Reshetikhin-like formula (off-shell BVs)
  - $Y(gl_3)$ [Reshetikhin '86]
  - $U_q(\widehat{gl}_3)$ [1311.3500, 1401.4355], $Y(gl_{2|1})$ [1605.09189]
  - $Y(gl_{\mathfrak{m}|\mathfrak{n}})$ [1704.08173], $U_q(\widehat{gl}_{\mathfrak{n}})$ [in preparation]
- Determinant form
  - On-shell BVs: $Y(gl_3)$ [1207.0956],
  - Semi-on-shell BVs: $Y(gl_{2|1})$ [1606.03573]
- Gaudin Determinant form (on-shell BVs)
  - $Y(gl_3)$ [Reshetikhin '86]
  - $Y(gl_{\mathfrak{m}|\mathfrak{n}})$ [1705.09219], $U_q(\widehat{gl}_{\mathfrak{n}})$ [in preparation]

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Summary (3/3): Form Factors

- Twisted scalar product trick
  - works for $Y(gl_{\mathfrak{m}|\mathfrak{n}})$ and $U_q(\widehat{gl}_{\mathfrak{n}})$
  - Determinant form
    - $Y(gl_3)$ [1211.3968]
    - $Y(gl_{2|1})$ [1607.04978]
- The zero mode method
  - works for $Y(gl_{\mathfrak{m}|\mathfrak{n}})$
  - Infinite Bethe roots for $U_q(\widehat{gl}_{\mathfrak{n}})$? [in progress]
  - Determinant form
    - $Y(gl_3)$ [1312.1488, 1406.5125]
    - $Y(gl_{2|1})$ [1607.04978]
- Coproduct formula
  - The coproduct formula exists for $Y(gl_{\mathfrak{m}|\mathfrak{n}})$ and $U_q(\widehat{gl}_{\mathfrak{n}})$
  - $Y(gl_3)$ for 2-component Bose gaz [1502.06749, 1503.00546]
  - $Y(gl_{2|1})$ for t-J model [Fuksa-Slavnov 1701.05866], [J. Fuksa 1611.00943]

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Conclusion: still a lot to do...

- A simpler expression for the scalar product of off-shell BVs
  - Possibly an integral representation
    - See for instance the work of M. Wheeler [1306.0552]
    - See also the integral representation coming from Projection method
  - A determinant form in the general case (including $U_q(\widehat{gl_3})$)
    - Note the determinant expression for XXX model in the thermodynamic limit by I. Kostov [1403.0358]
    - Note also for $Y(gl_N)$ with fundamental representations the approach by N. Grommov using a single 'B'-operator [1610.08032]
- Zero mode method ($U_q(\widehat{gl_\mathfrak{n}})$ case) [work in progress]
- Applications:
  - Multi-component Bose gas
  - tJ-model
  - SYM...
- Complete calculation of correlation functions, asymptotics, etc...

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Thank you!

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Happy B
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Jean-Michel !