Four point functions in 2D geometrical stat. mech. models

Work in (slow) progress with J. Jacobsen

There is no royal road to geometry (Euclid)
Geometrical correlations

- Are natural to consider in ordinary spin models such as the Ising model:

- Are even more natural to consider in geometrical models such as percolation or self-avoiding walks.

- Are also of interest in related contexts such as Anderson transitions.
the difficulty lies in the (mild) non-locality of the questions asked and objects considered can be dealt with but at a price: loss of unitarity
waves of theoretical approaches (in 2D)

- **Conformal invariance mixed with Coulomb gas techniques**
  - Critical exponents: eg fractal dimension of Ising clusters
  - A few correlation functions (partition functions): eg probability to have spin clusters of such and such shape on torus

- **Schramm Loewner Evolution**
  - Proofs of values of critical exponents
  - More correlation functions: eg probability of having clusters percolate through a rectangle with such and such shape

- **Liouville $c < 1$**
  - See Jacobsen’s talk
The forefront is now the **four point functions**

it is a non trivial step because Liouville \( c < 1 \) is **sick**:  
\[
\hat{C}(\hat{\alpha}_1, \hat{\alpha}_2, 0) \neq 0 \\
\hat{C}(\hat{\alpha}, 0, 0) \neq 0
\]

Virasoro degeneracies (and thus logarithmic features) crop up

Simplest example (probably) : in percolation problem

\[ P_{aabab} \]

\[ P_{aabb} \]

\[ P_{abba} \]

\[ P_{aaaa} \]
Reminder: Potts model = cluster model

\[ Z_{\text{Potts}} = \sum_{\{\sigma_i\}} \prod_{\langle jk \rangle} e^{K\delta(\sigma_i, \sigma_j)}, \quad \sigma_i = 1, \ldots, Q \]

\[ = \]

\[ Z_{\text{FK}} = \sum_c (e^K - 1)^B Q^C \]
Cluster are in one to one correspondence with loops:

\[ Z_{FK} = \sum_{c} (e^K - 1)^B Q^C \]

\[ Z_{Loops} = (\sqrt{Q})^S \sum_{P} (\sqrt{Q})^L \]
An interesting proposal [Picco, Ribault, Santachiara]

Conformal invariance restricts the form of four point functions of local fields. They can be written

$$R = \sum_{(\Delta, \bar{\Delta}) \in S^{(k)}} D_{\Delta, \bar{\Delta}}^{(k)} \mathcal{F}_{\Delta}^{(k)}(\{z_i\}) \mathcal{F}_{\bar{\Delta}}^{(k)}(\{\bar{z}_i\})$$

channel | limit
---|---
$s$ | $z_1 \to z_2$
$t$ | $z_1 \to z_4$
$u$ | $z_1 \to z_3$

The unknowns are the values of $\Delta, \bar{\Delta}$ (the spectrum). The functions $\mathcal{F}_{\Delta}^{(k)}$ are determined by general principles for generic values of $c$ and $\Delta$ (conformal blocks)

make a conjecture about the spectrum
assume the spectrum is the same in two of the three channels
solve crossing consistency conditions e.g.

$$\sum_{(\Delta, \bar{\Delta}) \in S} D_{\Delta, \bar{\Delta}} \left( \mathcal{F}_{\Delta}^{(s)}(\{z_i\}) \mathcal{F}_{\bar{\Delta}}^{(s)}(\{\bar{z}_i\}) - \mathcal{F}_{\Delta}^{(t)}(\{z_i\}) \mathcal{F}_{\bar{\Delta}}^{(t)}(\{\bar{z}_i\}) \right) = 0$$
The purpose of this talk: discuss and show how to determine spectras in s-channel of geometrical problems [Jacobsen, Saleur]

unpublished work
Some detail on exponents

Basic data

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<tr>
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$P_{abab} \propto |z_{13}z_{24}|^{-4\Delta} G_{abab}(z, \bar{z}), \quad z \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}}$ and $G_{abab}(z, \bar{z}) = \sum_{h, \bar{h} \in S} C_{\sigma} \Phi_h,\bar{h} C_{\Phi_h,\bar{h}} \sigma \mathcal{F}_h^{(s)}(z) \mathcal{F}_h^{(s)}(\bar{z})$
Some detail on exponents

- **Basic data**
  \[ \sqrt{Q} = 2 \cos \frac{\pi}{m+1}, \quad m \in [1, \infty] \]
  \[ c = 1 - \frac{6}{m(m + 1)} \quad \text{order parameter:} \quad \Delta \equiv h_{1/2,0} \]
  \[ h_{rs} = \frac{[(m + 1)r - ms]^2 - 1}{4m(m + 1)} \]

- **Conjectured spectra. Eg**

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\[ P_{abab} \propto |z_{13z24}|^{-4\Delta} G_{abab}(z, \bar{z}), \quad z = \frac{z_{12z34}}{z_{13z24}} \quad \text{and} \quad G_{abab}(z, \bar{z}) = \sum_{h,\bar{h} \in S} C_{\sigma\sigma}^{\Phi_{h,\bar{h}}} C_{\Phi_{h,\bar{h}}}^{(s)}(z) \bar{F}_h^{(s)}(\bar{z}) \]
what it means:

\[ G_{abab}(z, \bar{z}) \approx |z|^{-4\Delta} \sum_{h, \bar{h} \in S} C_{\sigma \sigma \Phi_{h, \bar{h}}} C_{\Phi_{h, \bar{h}}} z^h \bar{z}^{\bar{h}} (1 + O(z, \bar{z})) \]

\[
\begin{array}{cccc}
1 & s & 3 \\
2 & & 4 \\
\end{array}
\]

**how can we determine S?**

our work: a mixture of representation theory and numerics
The basic strategy

- Study problem on a cylinder

\[ P_{abab} \propto \sum_{h,\tilde{h} \in S} C_{\sigma\sigma F_{h,\tilde{h}}} C_{\Phi_{h,\tilde{h}}^\sigma} \left( 4 \sin^2 \frac{2\pi a}{L} \right)^{h+\tilde{h}} (-1)^{h-\tilde{h}} \xi^h \bar{\xi}^{\tilde{h}} (1 + O(\xi, \bar{\xi})) \quad \text{with} \quad \xi \equiv e^{-2\pi(l+ix)/L}, \quad \bar{\xi} \equiv e^{-2\pi(l-ix)/L} \]

\[ w_1 = ia, \quad w_2 = -ia \]
\[ w_3 = i(a + x) + l, \quad w_4 = i(-a + x) + l \]
The basic strategy

- Study problem on a cylinder

\[ P_{abab} \propto \sum_{h, \bar{h} \in \mathcal{S}} C_{\sigma \sigma_{h, \bar{h}}} C_{\Phi_{h, \bar{h}}} \left( \frac{4 \sin^2 \frac{2\pi a}{L}}{1 + O(\xi, \bar{\xi})} \right) (-1)^{h+\bar{h}} \xi^h \bar{\xi}^\bar{h} \]

the usual contribution from transfer matrix eigenvalues:

\[ \lambda^l e^{-iPx} \]

\[ \lambda = \exp \left[ -2\frac{\pi}{L}(h + \bar{h}) \right] \]

\[ P = \frac{2\pi}{L}(h - \bar{h}) \in \mathbb{Z} \]

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with \( \xi = e^{-2\pi(l+ix)/L} \), \( \bar{\xi} = e^{-2\pi(l-ix)/L} \)

the usual contribution from transfer matrix eigenvalues: \( \lambda e^{-iPx} \)

\[ \lambda = \exp \left[ -\frac{2\pi}{L} (h + \bar{h}) \right] \]

\[ P = \frac{2\pi}{L} (h - \bar{h}) \in \mathbb{Z} \]

the amplitude corrected by logarithmic mapping (from plane to cylinder)
Strategy is brutal:

- choose $L$
- determine for $L$ all possible eigenvalues of the transfer matrix
- determine which critical exponents are associated with them
- calculate the probabilities numerically to arbitrary precision for many values of the distance between the two sets of points (using exact enumeration and transfer matrix techniques)
- write $P = \sum C\lambda^l e^{iP_x}$
- invert the system to determine all coefficients of $\lambda^l e^{-iP_x}$ for all $\lambda, P$
- extract the amplitudes (estimated for a given $L$)
- do it for as many $L$ as possible
- extrapolate
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a crucial step whose answer is mostly algebraic
A little algebra:

The defining relations are easily checked by using isotopy ambient on the boundary of the diagram, see Fig. 2, where the points on each of the opposite sides. Multiplication for instance the second relation becomes

\[ [e_i, e_j] = 0, \quad |i - j| \geq 2 \]

with the well known graphical representation

\[
\begin{array}{c}
\hline
\quad \ldots \quad \circlearrowleft \quad \ldots \\
\hline
1 \quad i \quad i+1 \\
\end{array}
\]

\[ e_i \equiv e_i \mod L \]

for instance the second relation becomes

\[
\begin{array}{c}
\circlearrowleft \quad \circlearrowleft \\
\end{array}
\]

\[ = \quad \circlearrowleft \\
\]

\[
\begin{array}{c}
\circlearrowleft \quad \circlearrowleft \\
\end{array}
\]

\[ (\text{recall loops are cluster boundaries}) \]

All finite dimensional modules are classified [Martin, Saleur; Graham Lehrer]

(Note: because of the non locality of the problem it is not totally obvious a priori that this algebra is all that’s needed. But it is true. The reason is that the algebra encompasses all spin correlations for all Q integer. The decomposition of the transfer matrix on standard modules of TL is known for all Q)
Finite dimensional modules are the \( \mathcal{W}_{j, \frac{z^2}{z^2+1}} \), characterized by 2\( j \) through lines and the phase \( z = e^{iK}(z^{-1} = e^{-iK}) \) acquired by such a line when it winds clockwise (counterclockwise) around the cylinder.

Note that it is natural to ask \( e^{2iKj} = 1, \ j \neq 0 \) while when \( j = 0 \) non contractible loops get a fugacity \( n_{NC} = z + z^{-1} \).

The modules necessary to reproduce the probabilities are the same as those appearing in the torus partition function of the Potts model [DiFrancesco, Saleur, Zuber]

\[
\begin{align*}
\mathcal{W}_{0, q^2} \\
\mathcal{W}_{0, -1} \\
\mathcal{W}_{j, e^{2i \pi p/M}}, M | j.
\end{align*}
\]

this is all exact in finite size for all L
A little CFT:

\[
\mathcal{W}_{0,q^2} \quad \xrightarrow{=} \quad \bar{F}_{0,q^2} = \sum_{r=1}^{\infty} K_{r1}(q)K_{r1}(\bar{q})
\]

\[
\mathcal{W}_{0,-1} \quad \xrightarrow{=} \quad F_{0,-1} = \frac{q^{-c/24}\bar{q}^{-c/24}}{P(q)P(\bar{q})} \sum_{e \in \mathbb{Z}} q^{h_{e+1/2,0}}\bar{q}^{h_{e+1/2,0}}
\]

\[
\mathcal{W}_{j,e^{2i\pi p/M}} \quad \xrightarrow{=} \quad F_{j,e^{2i\pi p/M}} = \frac{q^{-c/24}\bar{q}^{-c/24}}{P(q)P(\bar{q})} \sum_{e \in \mathbb{Z}} q^{h_{e+\frac{p}{M}j}}\bar{q}^{h_{e+\frac{p}{M}j}}, \; M|j, \; j \text{ integer}
\]

Note how \( S_{\mathbb{Z}+\frac{1}{2},2\mathbb{Z}} \) is part of this.
in the s-channel. We find in fact \( S_{\mathbb{Z} + \frac{p}{N}, 2n} \); \( n > 0 \), \( 2np/N \) an odd integer.

This contains the odd spin fields in \( S_{\mathbb{Z} + \frac{1}{2}, 2\mathbb{Z}} \)

but also those in \( S_{\mathbb{Z} + \frac{1}{4}, 4\mathbb{Z}} \) etc etc!
Amplitude ratios as of today (extraordinarily time consuming calculation)

\[ P_{aabb} - P_{abba} \propto (z\bar{z})^{-2h_{1/2,0}} \left( A_{\Phi h_{1/2,-2,h_{1/2,2}}} z^{h_{1/2,-2}} \bar{z}^{h_{1/2,2}} + A_{\Phi h_{3/2,-2,h_{3/2,2}}} z^{h_{3/2,-2}} \bar{z}^{h_{3/2,2}} + \ldots \right) \]

two poles
vanishes exactly at Q=3,4
non zero otherwise
Amplitude ratios as of today (extraordinarily time consuming calculation)

\[
P_{aabb} - P_{abba} \propto (z \tilde{z})^{-2h_{1/2,0}} \left( A_{\Phi h_{1/2,-2},h_{1/2,2}} z^{h_{1/2,-2}} \tilde{z}^{h_{1/2,2}} + A_{\Phi h_{3/2,-2},h_{3/2,2}} z^{h_{3/2,-2}} \tilde{z}^{h_{3/2,2}} + \ldots \right)
\]

two poles
vanishes exactly at \( Q=3,4 \)
non zero otherwise

\( (2 \cos \frac{3\pi}{8})^2 = 0.5858 \)

\( (2 \cos \frac{\pi}{8})^2 = 3.4142 \)

Jordan cells of \( L_0 \) at these two values
A technical note: the convergence of the amplitudes is little explored. An example:

For $Q=2$

$$G_{\alpha\alpha\alpha\alpha} = P_{aaaa} + P_{aabb} + P_{abba} + P_{abab}$$

where

$$G_{\alpha\alpha\alpha\alpha} = \langle \prod_{i=1}^{4} (Q \delta_{\sigma_i,\alpha} - 1) \rangle$$

is our usual 4 point spin correlator

Expansion in the s-channel:

$$1 + \frac{1}{4} z^{1/2} \bar{z}^{1/2} + \frac{1}{16} (z^{1/2} \bar{z}^{3/2} + z^{3/2} \bar{z}^{1/2}) + \frac{1}{64} z^{3/2} \bar{z}^{3/2} + \frac{1}{64} (z^2 + \bar{z}^2) + \ldots$$

ratio of measured over expected vs $1/L$ for $a=L/2$
Conclusions

“Experimental” analysis of correlators in non-unitary CFTs seems the best way to make progress. Many question can be explored, many checks performed.

The algebraic framework allows one to predict some of the structure of the coupling constants: eg the poles seen earlier.

It looks like the nice prediction of [Picco, Ribault, Santachiara] is not correct.

In principle knowing the spectrum we could determine the full correlators by bootstrap. But:

- the spectrum is extremely rich with exponents close to each other
- the spectrum involves many degenerate fields, even for $Q$ generic. The naive treatment of Zamolodchikov’s conformal blocks at these values of $\Delta$ does not seem to be enough

Can we use any of the form-factors/QISM formalism here?

work in progress
Thank you and happy birthday Jean Michel