Four point functions in 2D geometrical stat. mech. models

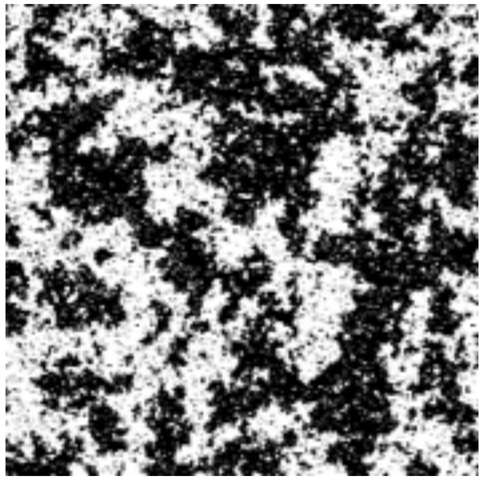
Work in (slow) progress with J. Jacobsen

There is no royal road to *geometry* (Euclid)

Geometrical correlations

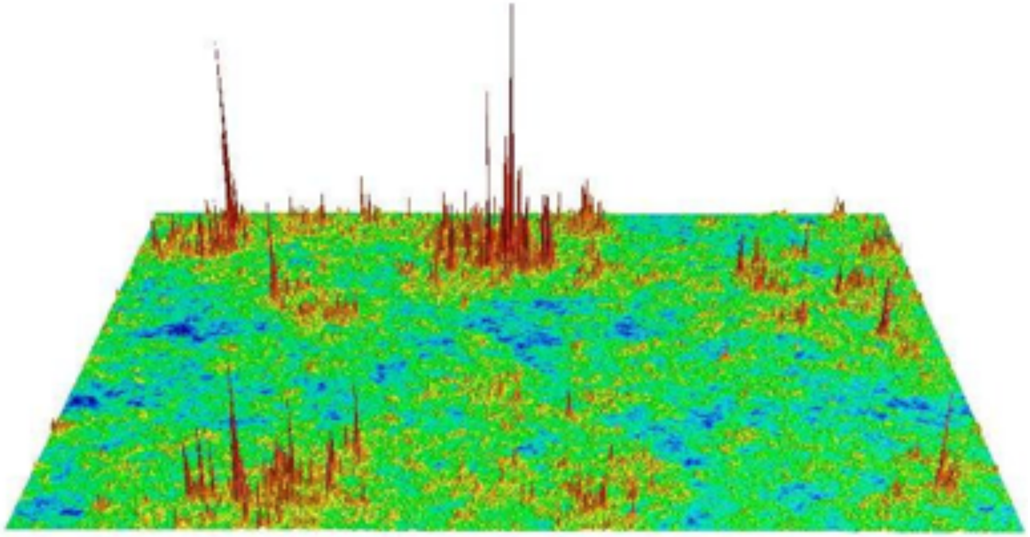
- Are natural to consider in ordinary spin models such as Ising model:

spin clusters at criticality

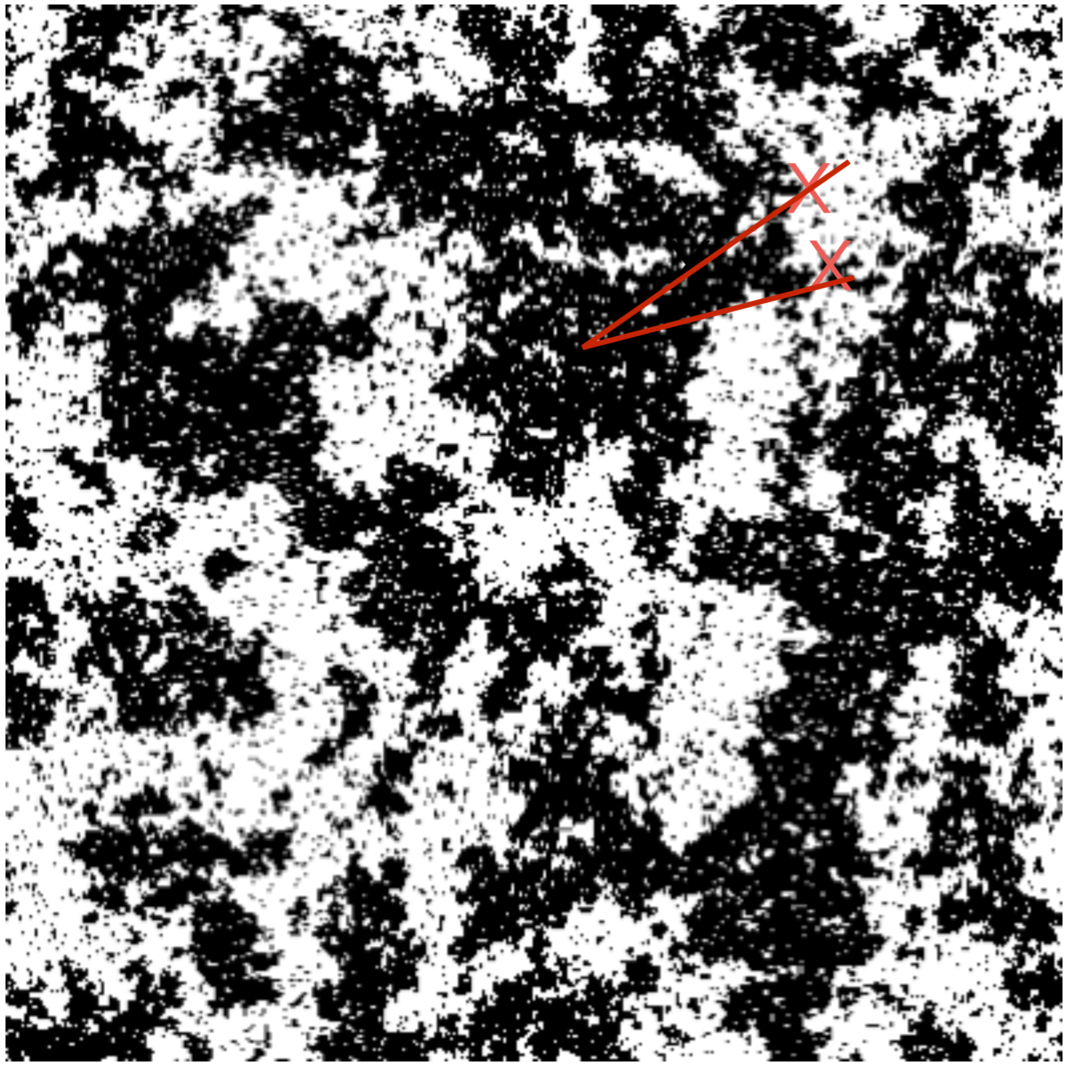


- Are even more natural to consider in geometrical models such as percolation or self-avoiding walks

- Are also of interest in related contexts such as Anderson transitions



the difficulty lies in the (mild) non-locality of the questions asked and objects considered



can be dealt with but at a price:
loss of unitarity

waves of theoretical approaches (in 2D)

- Conformal invariance mixed with Coulomb gas techniques

→ critical exponents: eg fractal dimension of Ising clusters

a few correlation functions (partition functions): eg probability to have spin clusters of such and such shape on torus

- Schramm Loewner Evolution

→ proofs of values of critical exponents

more correlation functions: eg probability of having clusters percolate through a rectangle with such and such shape

- Liouville $c < 1$

→ see Jacobsen's talk

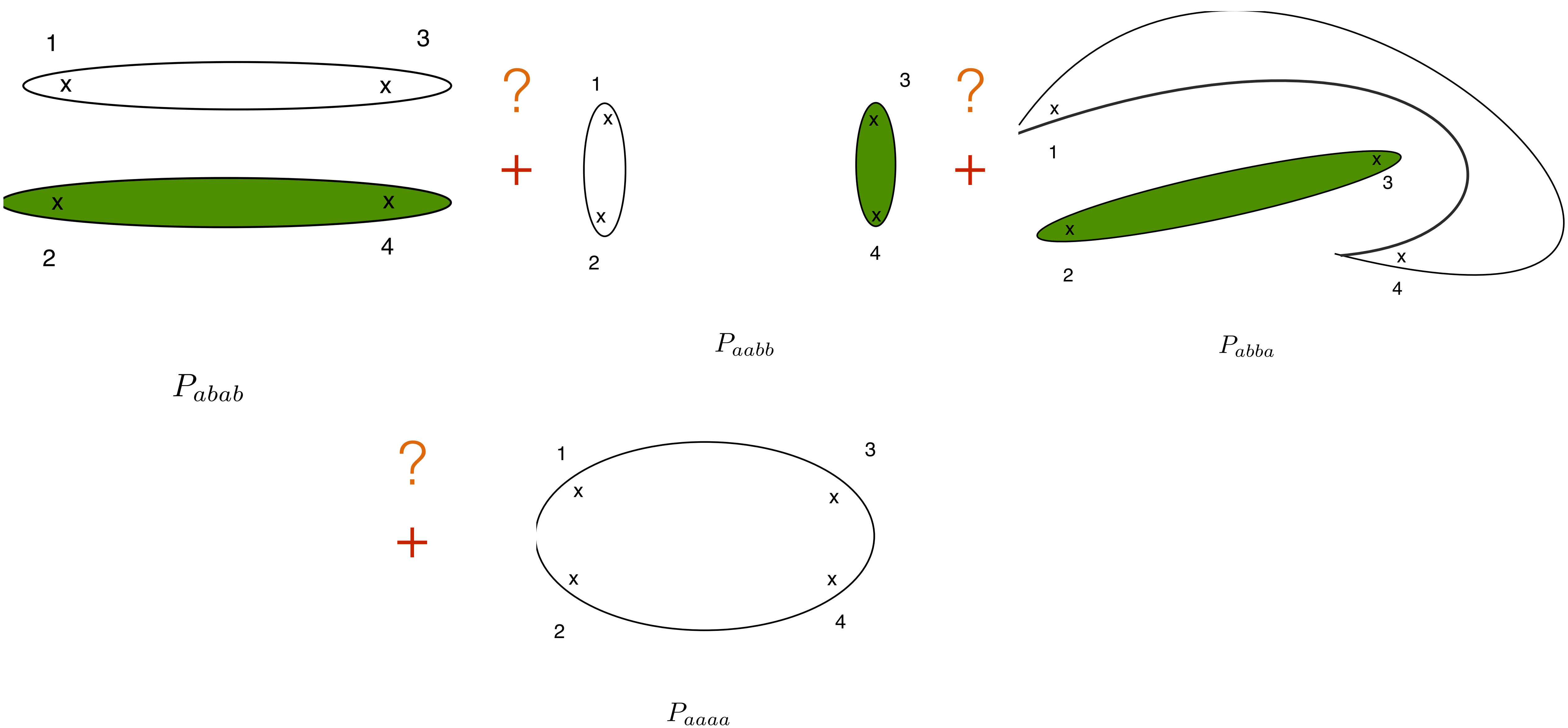
The forefront is now the four point functions

it is a non trivial step because Liouville $c<1$ is sick:

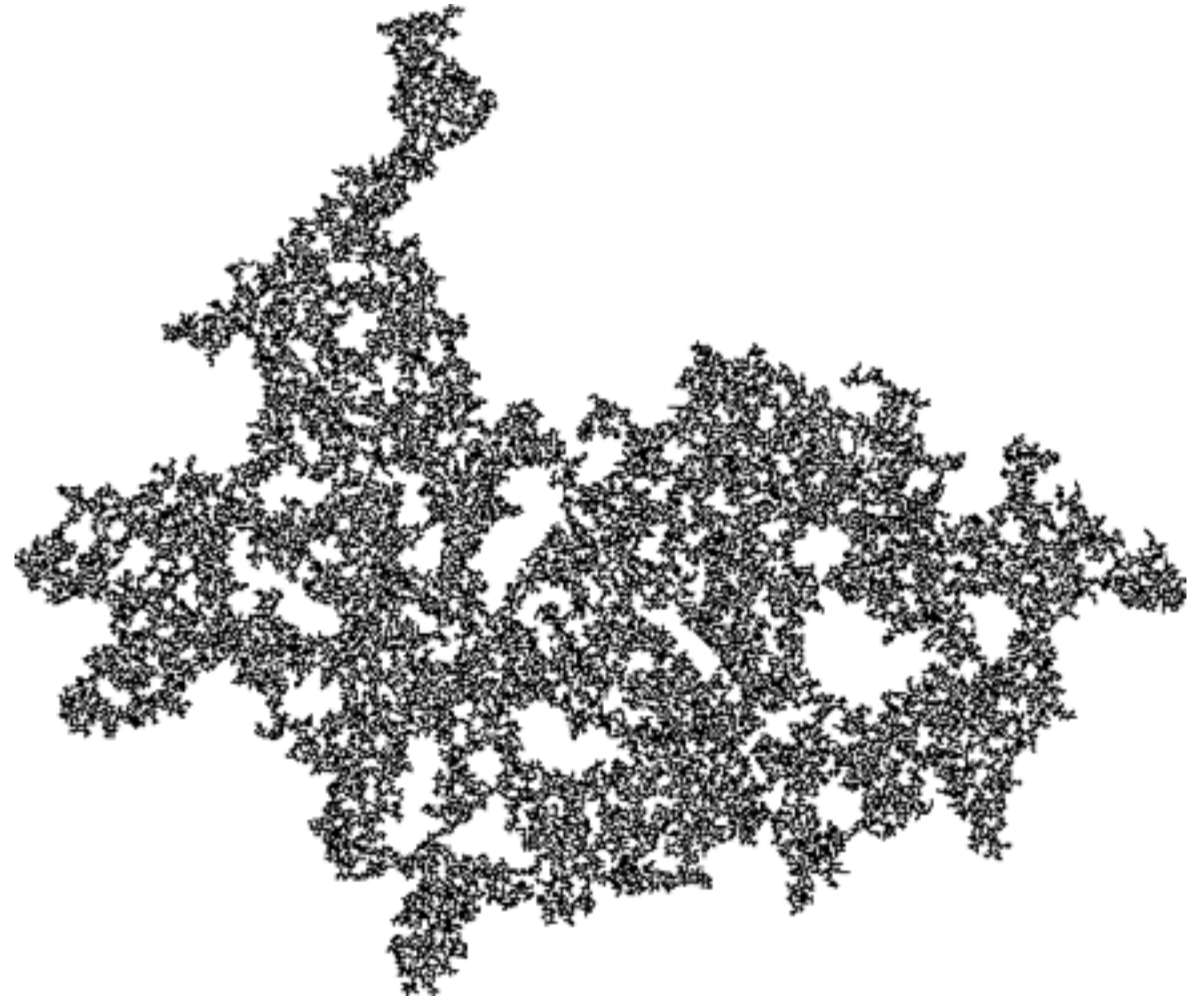
$$\hat{C}(\hat{\alpha}_1,\hat{\alpha}_2,0)\neq 0$$
$$\hat{C}(\hat{\alpha},0,0)\neq 0$$

Virasoro degeneracies (and thus logarithmic features) crop up

→ Simplest example (probably) : in percolation problem



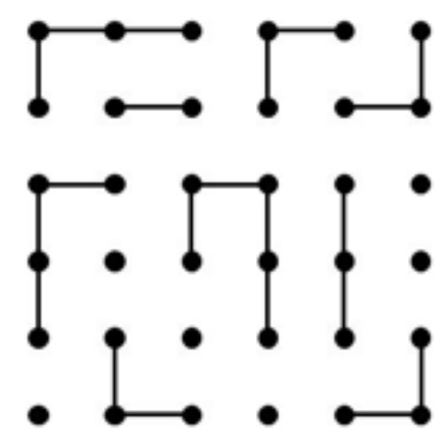
Reminder: Potts model = cluster model



$$Z_{\mathrm{Potts}} = \sum_{\{\sigma_i\}} \prod_{<jk>} e^{K\delta(\sigma_i,\sigma_j)}, \quad \sigma_i = 1, \ldots, Q$$

$=$

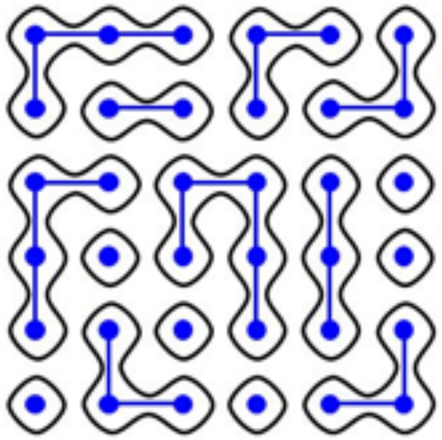
$$Z_{\mathrm{FK}} = \sum_{\mathcal{C}} (e^K - 1)^B Q^C$$



CLuster are in one to one correspondence with loops:

$$Z_{\mathrm{FK}} = \sum_{\mathcal{C}} (e^K - 1)^B Q^C$$

$$Z_{\mathrm{Loops}} = (\sqrt{Q})^S \sum_{\mathcal{P}} (\sqrt{Q})^L$$

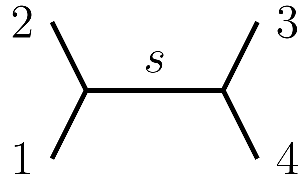


An interesting proposal [Picco,Ribault, Santachiara]

Conformal invariance restricts the form of four point functions of local fields. They can be written

$$R = \sum_{(\Delta,\bar{\Delta})\in\mathcal{S}^{(k)}} D^{(k)}_{\Delta,\bar{\Delta}} \mathcal{F}^{(k)}_{\Delta}(\{z_i\})\mathcal{F}^{(k)}_{\bar{\Delta}}(\{\bar{z}_i\}) \quad , \quad (k \in \{s,t,u\}) \ .$$

channel	limit
s	$z_1 \to z_2$
t	$z_1 \to z_4$
u	$z_1 \to z_3$



The unknowns are the values of $\Delta,\bar{\Delta}$ (the spectrum). The functions $\mathcal{F}^{(k)}_{\Delta}$ are determined by general principles for generic values of c and Δ (conformal blocks)

make a conjecture about the spectrum
assume the spectrum is the same in two of the three channels
solve crossing consistency conditions e.g.

$$\sum_{(\Delta,\bar{\Delta})\in\mathcal{S}} D_{\Delta,\bar{\Delta}} \left(\mathcal{F}^{(s)}_{\Delta}(\{z_i\})\mathcal{F}^{(s)}_{\bar{\Delta}}(\{\bar{z}_i\}) - \mathcal{F}^{(t)}_{\Delta}(\{z_i\})\mathcal{F}^{(t)}_{\bar{\Delta}}(\{\bar{z}_i\})\right) = 0$$

- The purpose of this talk: discuss and show how to determine spectras in s-channel of geometrical problems [Jacobsen, Saleur]

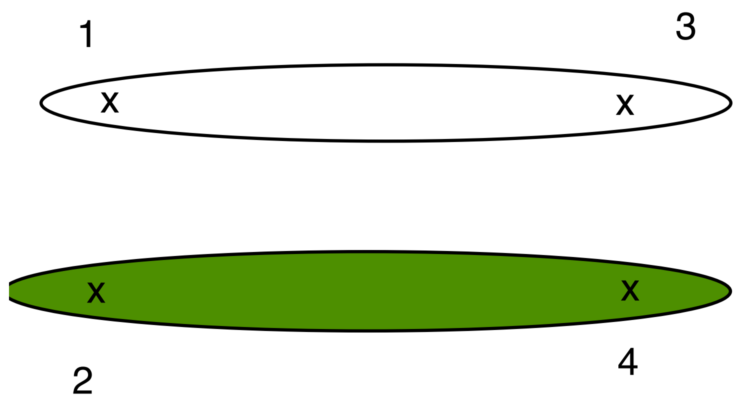
unpublished work

Some detail on exponents

- Basic data $$\sqrt{Q} = 2\cos\frac{\pi}{m+1},\ m \in [1,\infty]$$

$$c = 1 - \frac{6}{m(m+1)} \qquad h_{rs} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}$$

order parameter: $$\Delta \equiv h_{1/2,0}$$



P_{abab}

- Conjectured spectra. Eg

	s	t	u
P_{aabb}	$\mathcal{S}_0$	$\mathcal{S}_{\mathbb{Z}+1/2,2\mathbb{Z}}$	$\mathcal{S}_{\mathbb{Z}+1/2,2\mathbb{Z}}$
P_{abab}	$\mathcal{S}_{\mathbb{Z}+1/2,2\mathbb{Z}}$	$\mathcal{S}_{\mathbb{Z}+1/2,2\mathbb{Z}}$	$\mathcal{S}_0$
P_{abba}	$\mathcal{S}_{\mathbb{Z}+1/2,2\mathbb{Z}}$	$\mathcal{S}_0$	$\mathcal{S}_{\mathbb{Z}+1/2,2\mathbb{Z}}$

$$P_{abab} \propto |z_{13}z_{24}|^{-4\Delta} G_{abab}(z,\bar{z}), \qquad z \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}} \quad \text{and} \quad G_{abab}(z,\bar{z}) = \sum_{h,\bar{h}\in\mathcal{S}} C_{\sigma\sigma\Phi_{h,\bar{h}}} C_{\Phi_{h,\bar{h}}\sigma\sigma} \mathcal{F}_h^{(s)}(z) \overline{\mathcal{F}}_{\bar{h}}^{(s)}(\bar{z})$$

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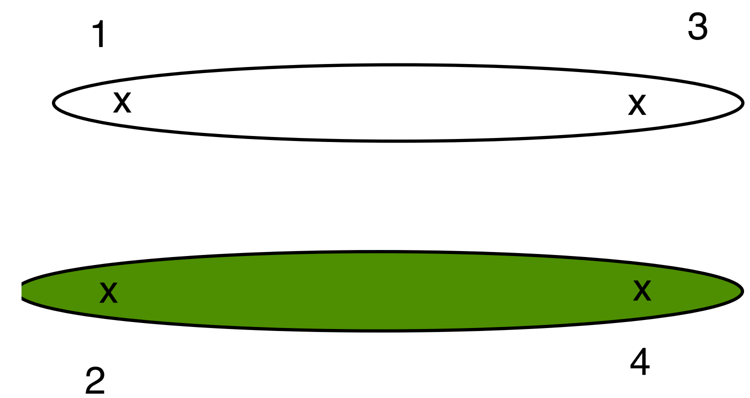

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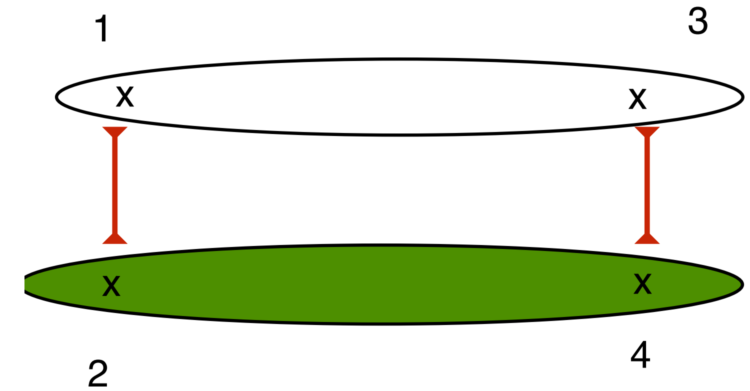
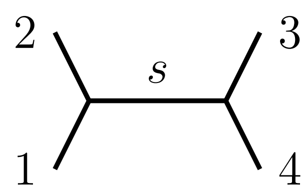
what it means:

$$G_{abab}(z,\bar{z}) \approx |z|^{-4\Delta} \sum_{h,\bar{h}\in\mathcal{S}} C_{\sigma\sigma\Phi_{h,\bar{h}}} C_{\Phi_{h,\bar{h}}\sigma} z^{h} \bar{z}^{\bar{h}} (1 + O(z,\bar{z}))$$





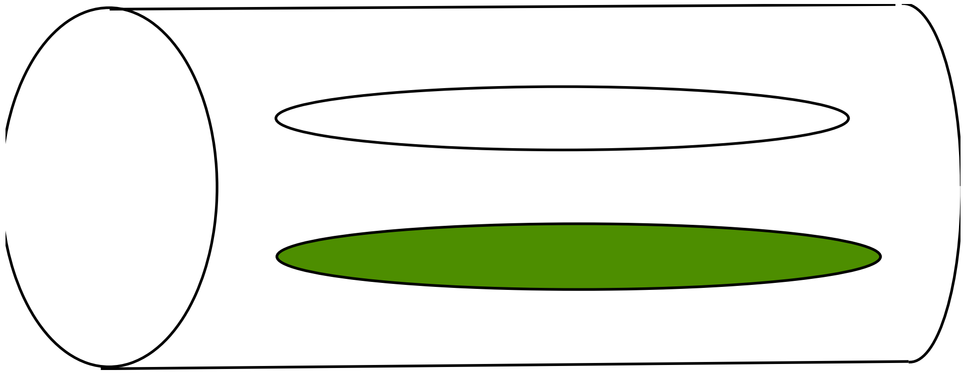
P_{abab}

how can we determine S?

our work: a mixture of representation theory and numerics

The basic strategy

- Study problem on a cylinder



$$w_1 = ia, \; w_2 = -ia$$
$$w_3 = i(a+x)+l, \; w_4 = i(-a+x)+l$$

$$P_{abab} \propto \sum_{h,\bar{h} \in \mathcal{S}} C_{\sigma\sigma\Phi_{h,\bar{h}}} C_{\Phi_{h,\bar{h}}\sigma} \left(4\sin^2\frac{2\pi a}{L}\right)^{h+\bar{h}} (-1)^{h-\bar{h}} \xi^h \bar{\xi}^{\bar{h}} (1 + O(\xi,\bar{\xi}))$$

with $\xi \equiv e^{-2\pi(l+ix)/L}, \; \bar{\xi} \equiv e^{-2\pi(l-ix)/L}$

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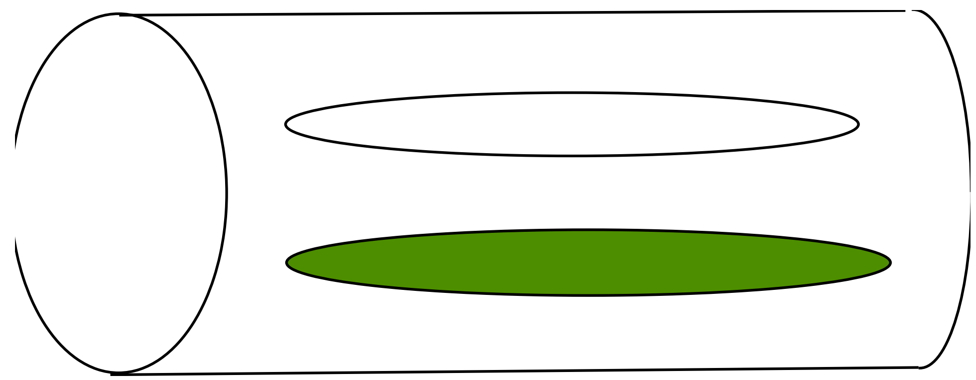

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the usual contribution from
transfer matrix eigenvalues : $\lambda^l e^{-iPx}$

$$\lambda = \exp\left[-2\frac{\pi}{L}(h+\bar{h})\right]$$

$$P = \frac{2\pi}{L}(h-\bar{h}) \in \mathbb{Z}$$

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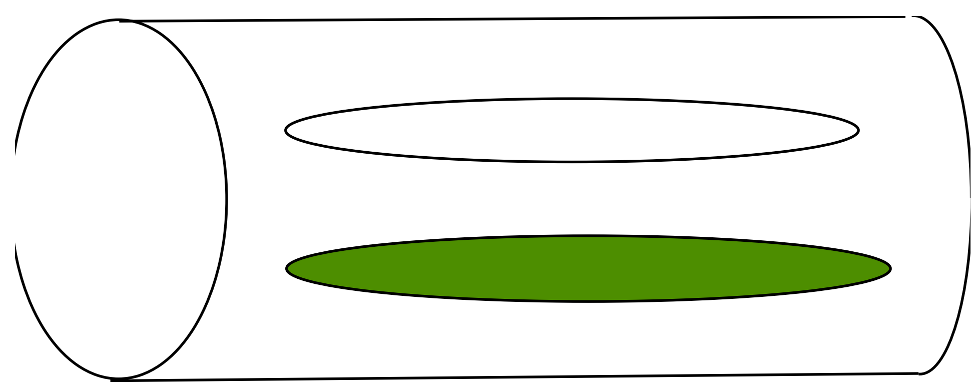

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the amplitude corrected by logarithmic mapping (from plane to cylinder)

Strategy is brutal:

- choose L
- determine for L all possible eigenvalues of the transfer matrix
- determine which critical exponents are associated with them
- calculate the probabilities numerically to arbitrary precision for many values of the distance between the two sets of points (using exact enumeration and transfer matrix techniques)
- write $P = \sum C \lambda^l e^{iPx}$
- invert the system to determine all coefficients of $\lambda^l e^{-iPx}$ for all λ, P
- extract the amplitudes (estimated for a given L)
- do it for as many L as possible
- extrapolate

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a crucial step whose answer is mostly algebraic

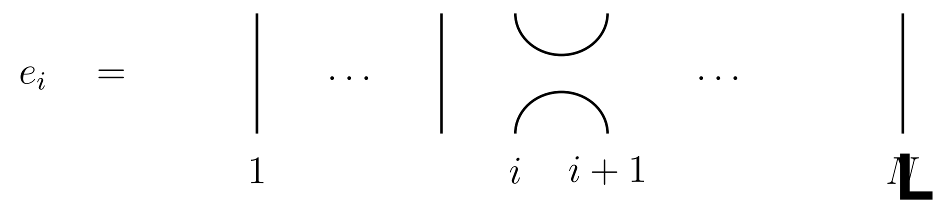
A little algebra:

the basic object is the
affine Temperley Lieb algebra

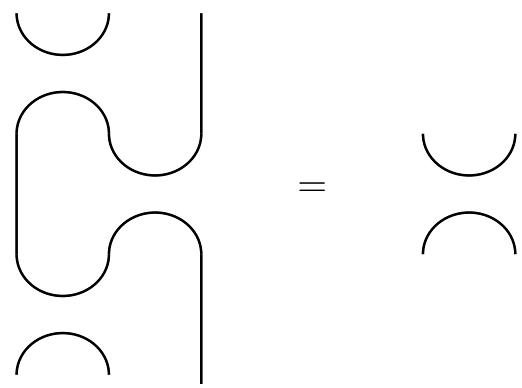
$$
\begin{aligned}
e_i^2 &= \sqrt{Q} e_i \\
e_i e_{i\pm 1} e_i &= e_i \\
[e_i, e_j] &= 0, \ |i-j| \geq 2
\end{aligned}
$$

$$e_i \equiv e_{i \text{ mod L}}$$

with the well known
graphical representation



for instance the second relation
becomes



(recall loops are cluster boundaries)

All finite dimensional modules are classified [Martin, Saleur; Graham Lehrer]

(Note: because of the non locality of the problem it is not totally obvious a priori that this algebra is all that's needed. But it is true. The reason is that the algebra encompasses all spin correlations for all Q integer. The decomposition of the transfer matrix on standard modules of TL is known for all Q)

Finite dimensional modules are the $\mathcal{W}_{j,z^2=\mathrm{e}^{2iK}}$ characterized by 2j through lines and the phase

$z = e^{iK}(z^{-1} = e^{-iK})$ acquired by such a line when it winds clockwise (counterclockwise) around the cylinder.

Note that it is natural to ask $e^{2iKj} = 1, \quad j \neq 0$ while when $j = 0$ non contractible loops get a fugacity

$n_{\mathrm{NC}} = z + z^{-1}$

The modules necessary to reproduce the probabilities are the same as those appearing in the torus partition function of the Potts model [DiFrancesco, Saleur, Zuber]

$$\overline{\mathcal{W}}_{0,\mathfrak{q}^2}$$

$$\mathcal{W}_{0,-1}$$

$$\mathcal{W}_{j,\mathrm{e}^{2i\pi p/M}}, M|j.$$

this is all exact in finite size for all L

A little CFT:

$$\overline{\mathcal{W}}_{0,\mathfrak{q}^2} \longmapsto \bar{F}_{0,\mathfrak{q}^2} = \sum_{r=1}^{\infty} K_{r1}(q) K_{r1}(\bar{q})$$

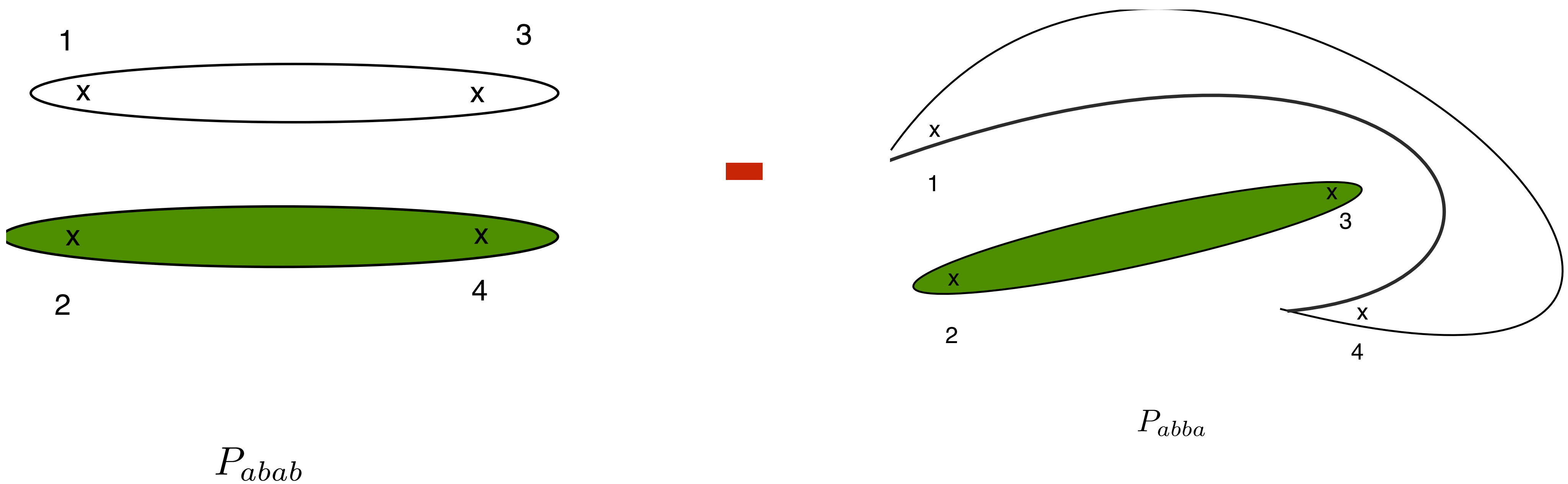
$$\mathcal{W}_{0,-1} \longmapsto F_{0,-1} = \frac{q^{-c/24}\bar{q}^{-c/24}}{P(q)P(\bar{q})} \sum_{e\in\mathbb{Z}} q^{h_{e+1/2,0}} \bar{q}^{h_{e+1/2,0}}$$

$$\mathcal{W}_{j,\mathrm{e}^{2i\pi p/M}} \longmapsto F_{j,\mathrm{e}^{2i\pi p/M}} = \frac{q^{-c/24}\bar{q}^{-c/24}}{P(q)P(\bar{q})} \sum_{e\in\mathbb{Z}} q^{h_{e+\frac{p}{M},-j}} \bar{q}^{h_{e+\frac{p}{M},j}}, \quad M|j, \quad j \text{ integer}$$

Note how $\mathcal{S}_{\mathbb{Z}+\frac{1}{2},2\mathbb{Z}}$ is part of this.

Results

- Focus on



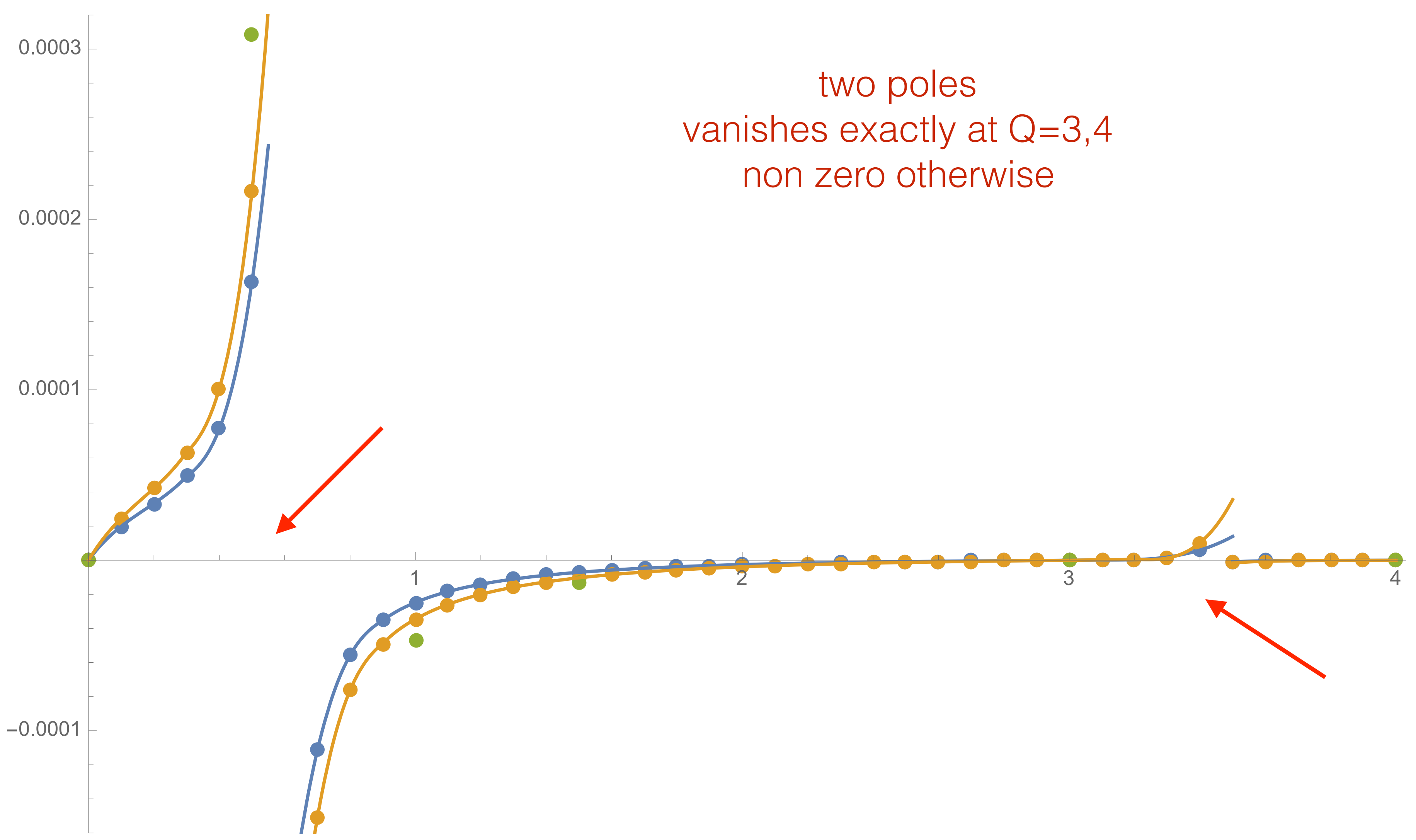
in the s-channel. We find in fact $\mathcal{S}_{\mathbb{Z}+\frac{p}{N},2n}; \;\; n>0, \;\; 2np/N$ an odd integer.

This contains the odd spin fields in $\mathcal{S}_{\mathbb{Z}+\frac{1}{2},2\mathbb{Z}}$

but also those in $\mathcal{S}_{\mathbb{Z}+\frac{1}{4},4\mathbb{Z}}$ etc etc!

Amplitude ratios as of today (extraordinarily time consuming calculation)

$$P_{aabb} - P_{abba} \propto (z\bar{z})^{-2h_{1/2,0}} \Big(A_{\Phi_{h_{1/2,-2},h_{1/2,2}}} z^{h_{1/2,-2}} \bar{z}^{h_{1/2,2}} + A_{\Phi_{h_{3/2,-2},h_{3/2,2}}} z^{h_{3/2,-2}} \bar{z}^{h_{3/2,2}} + \ldots \\ A_{\Phi_{h_{1/4,-4},h_{1/4,4}}} z^{h_{1/4,-4}} \bar{z}^{h_{1/4,4}} + \ldots \Big)$$



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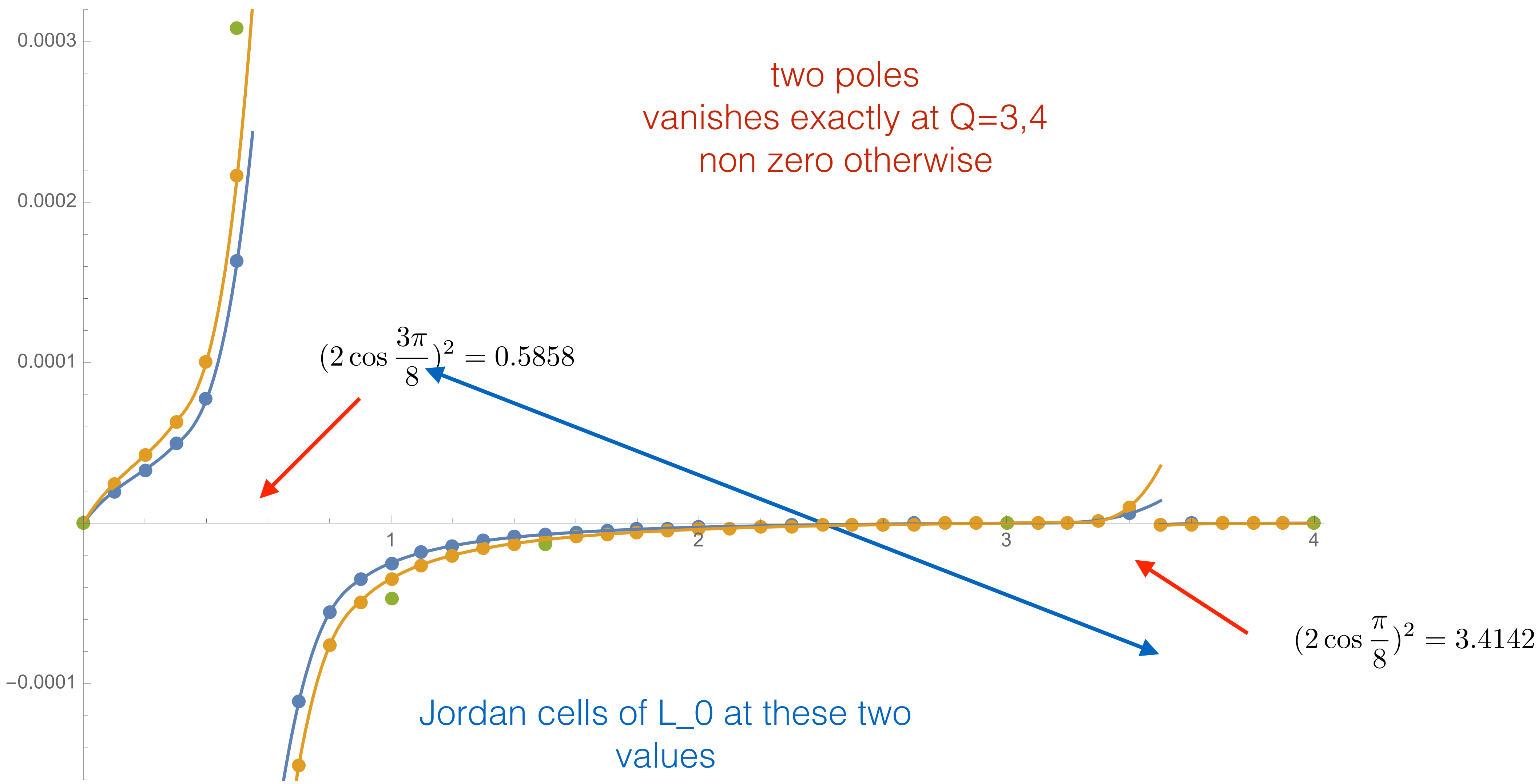
$$A_{\Phi_{h_{1/4,-4},h_{1/4,4}}} z^{h_{1/4,-4}} \bar{z}^{h_{1/4,4}} + \dots \Big)$$



A technical note: the convergence of the amplitudes is little explored. An example:

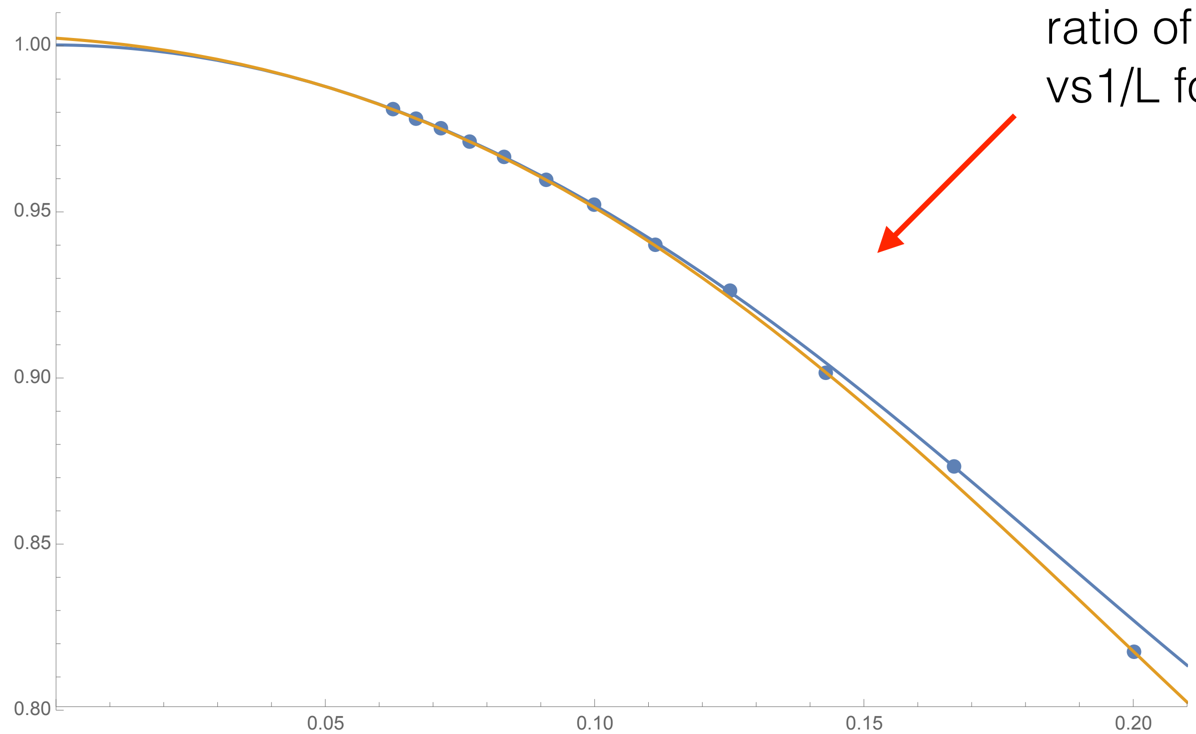
For Q=2

$$G_{\alpha\alpha\alpha\alpha} = P_{aaaa} + P_{aabb} + P_{abba} + P_{abab}$$

where $G_{\alpha\alpha\alpha\alpha} = \langle \prod_{i=1}^{4} (Q\delta_{\sigma_i,\alpha} - 1) \rangle$ is our usual 4 point spin correlator

Expansion in the s-channel:

$$1 + \frac{1}{4} z^{1/2} \bar{z}^{1/2} + \frac{1}{16}(z^{1/2}\bar{z}^{3/2} + z^{3/2}\bar{z}^{1/2}) + \frac{1}{64} z^{3/2}\bar{z}^{3/2} + \frac{1}{64}(z^2 + \bar{z}^2) + \ldots$$



ratio of measured over expected vs1/L for a=L/2

Conclusions

- "Experimental" analysis of correlators in non-unitary CFTs seems the best way to make progress
Many question can be explored, many checks performed

- The algebraic framework allows on to predict some of the structure of the coupling constants: eg the poles seen earlier

- It looks like the nice prediction of [Picco,Ribault, Santachiara] is not correct.

- In principle knowing the spectrum we could determine the full correlators by bootstrap.
But:
    - the spectrum is extremely rich with exponents close to each other
    - the spectrum involves many degenerate fields, even for Q generic. The naive treatment of Zamolodchikov's conformal blocks at these values of Δ does not seem to be enough

- Can we use any of the form-factors/QISM formalism here?

work in progress

Thank you and happy birthday Jean Michel