Correlation functions in N=4 SYM from integrability

w/ Y. Jiang, S. Komatsu, I. Kostov, arXiv:1604.03575
I. Kostov, D. S. in progress

Correlation functions of quantum integrable systems and beyond

23-26 October 2017, Lyon

AdS/CFT integrability
(Integrability in Gauge and String Theory)

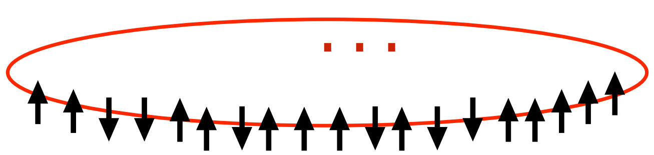
Integrability is relevant to the study of four-dimensional gauge theories:

- high energy QCD **[Lipatov; Faddeev, Korchemsky, 93-95]**
- supersymmetric large N theories **[Minahan, Zarembo, 02]**

One can learn a lot about the string duals of the gauge theories, also integrable **[Bena, Polchinski, Roiban, 02]**

Integrability is the unique non-perturbative tool giving access to the AdS/CFT correspondence **[Maldacena; Polyakov; Witten, ~98]** at **arbitrary coupling**

from **spin chains** to **strings**



Integrability data in planar $\mathcal{N} = 4$ SYM

Gauge invariant operators (traces over the gauge group SU(N)) in planar $\mathcal{N}$=4 SYM theory are mapped to states of a periodic integrable spin chain/field theory with psu(2,2|4) symmetry

[Minahan, Zarembo, 02]
[Beisert, Staudacher, 03, …]

Ferromagnetic vacuum: $\mathrm{Tr}\ Z^L$

$$Z = \Phi_1 + i\Phi_2$$
$$X = \Phi_3 + i\Phi_4$$

Excitations (magnons) in centrally extended psu(2|2) x psu(2|2) : **[Beisert, 05-06]**

Parametrised in terms of a rapidity variable with a square root branch cut

$$\frac{u}{2g} = x + \frac{1}{x} \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

Energy and momentum:

$$E(u) = \frac{i}{x^+} - \frac{i}{x^-}$$
$$p(u) = \frac{1}{i}\log\frac{x^+}{x^-} \qquad x^\pm = x(u \pm i/2)$$

Integrability data in planar $\mathcal{N} = 4$ SYM

the spin chain is secretly a field theory

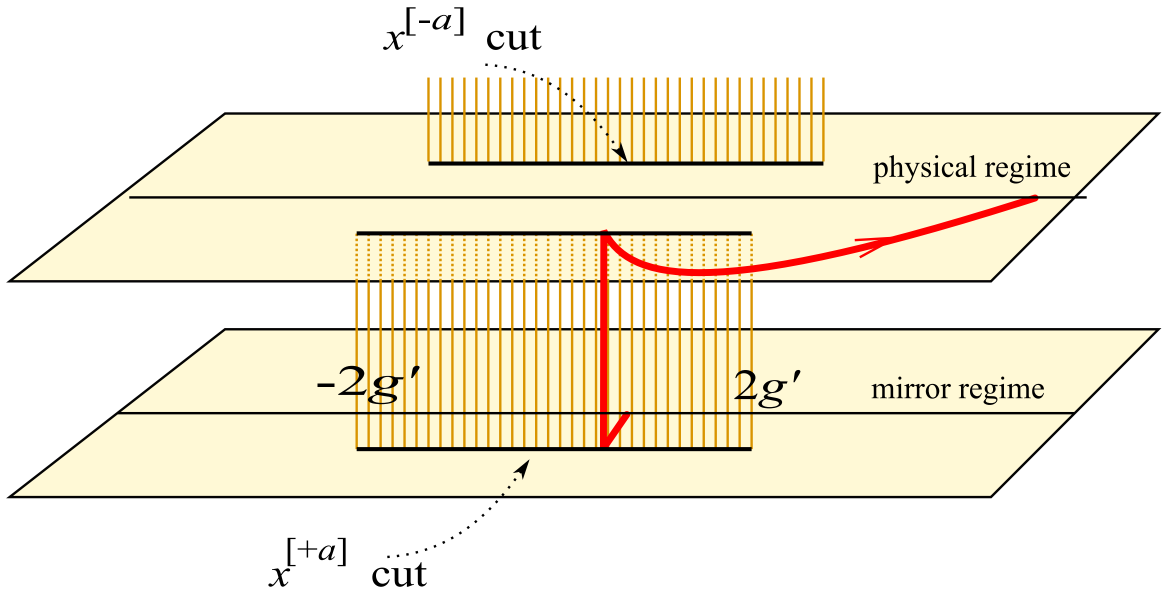
Usual rapidity representation for relativistic integrable field theories:

$$E(\theta) = M \cosh\theta$$
$$p(\theta) = M \sinh\theta$$

crossing (particle-hole) transformation $\theta \to \theta + i\pi$ $\quad (x,t) \to (-x,-t)$

mirror (virtual particle) transformation $\theta \to \theta + i\pi/2$ $\quad (x,t) \to (it, ix)$

$\mathcal{N}$=4 SYM rapidity space mirror transformation: u^γ



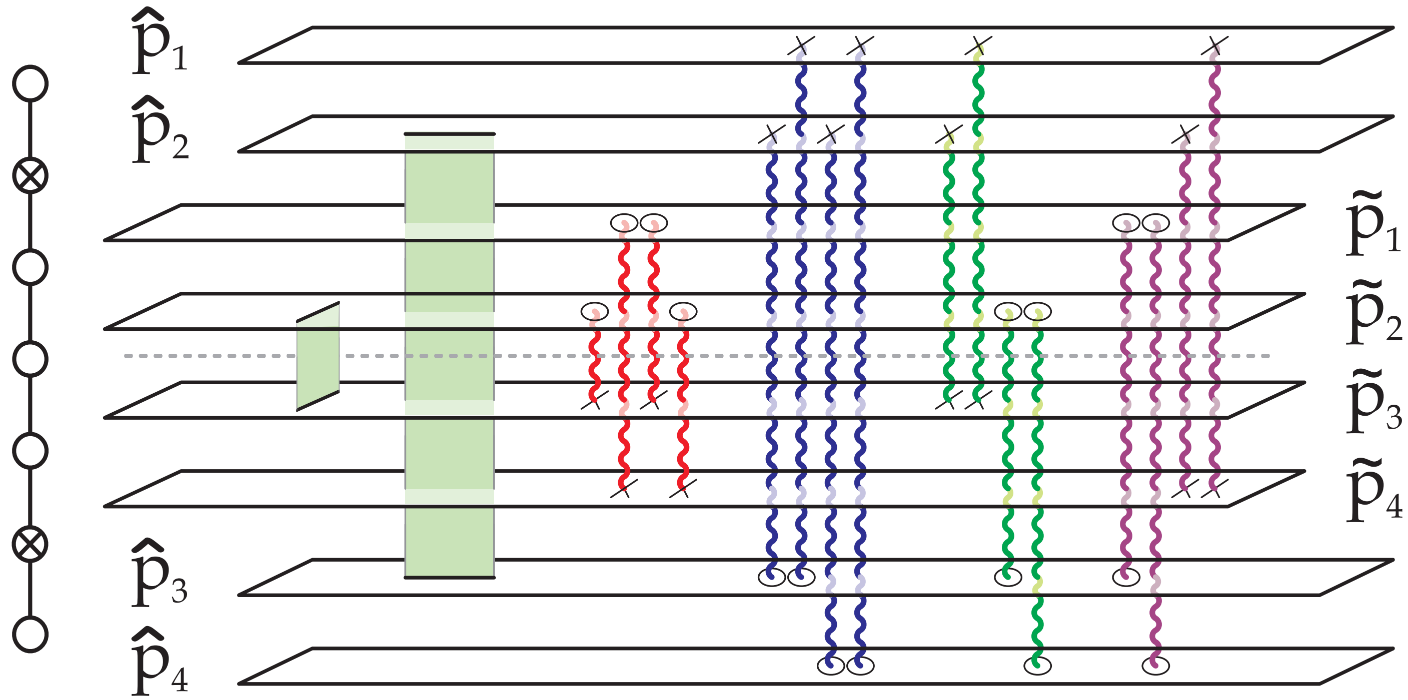
$$x^{[+a]} \to 1/x^{[+a]}$$
$$x^{[-a]} \to x^{[-a]}$$

mirror particles are responsible for finite-size corrections (TBA, Quantum Spectral Curve)

[Ambjorn, Janik, Kristjansen, 05,…] [Gromov, Kazakov, Vieira, 09,…]

Integrability data in planar $\mathcal{N} = 4$ SYM

Full information about a particular state is encoded in a 8-sheeted curve (quasi-momentum)

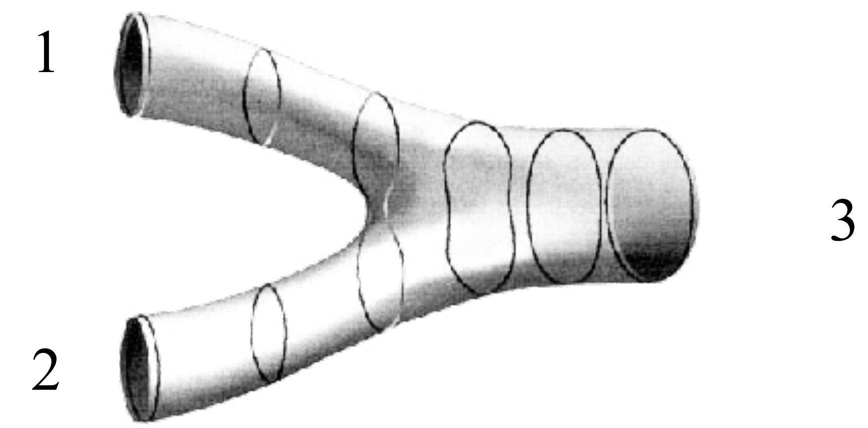


$\widetilde{p}_1, \ \widetilde{p}_2, \ \widetilde{p}_3, \ \widetilde{p}_4$ motion in S5

$\widehat{p}_1, \ \widehat{p}_2, \ \widehat{p}_3, \ \widehat{p}_4$ motion in AdS5

Correlation functions in $\mathcal{N}$=4 SYM

the three point function
dual to three-string interaction
is the basic building block for correlation function



$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}(\lambda)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

initial data: three states with definite conformal dimensions and psu(2,2|4) charges

$$\mathcal{O}_\alpha(x)\ , \quad \alpha = 1,2,3$$

each characterised by a set of rapidities $\mathbf{u}_\alpha$

and polarisations (or global rotations with respect to some reference BPS state, e.g Tr Z^L)

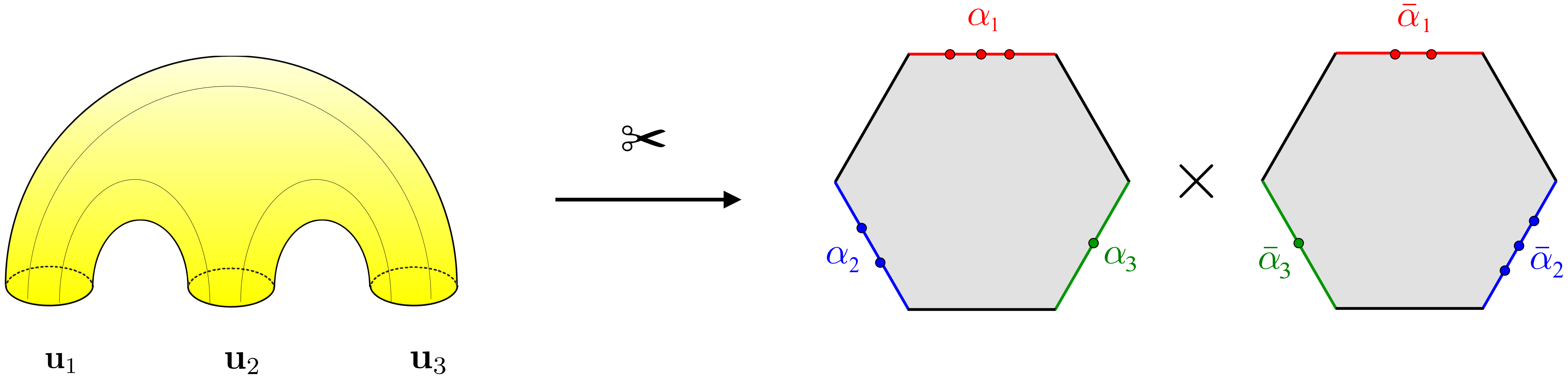
$$g_\alpha = e^{\zeta_\alpha^A J^A}$$

The hexagon decomposition

[Basso, Komatsu, Vieira, 15]

the asymptotic part of the three point function can be written as a sum over partitions for the three groups of rapidities $\mathbf{u}_1 = \alpha_1 \cup \bar{\alpha}_1, \mathbf{u}_2 = \alpha_2 \cup \bar{\alpha}_2, \mathbf{u}_3 = \alpha_3 \cup \bar{\alpha}_3$

similarity to the scalar product decomposition **[Izergin, Korepin, Slavnov,…]**



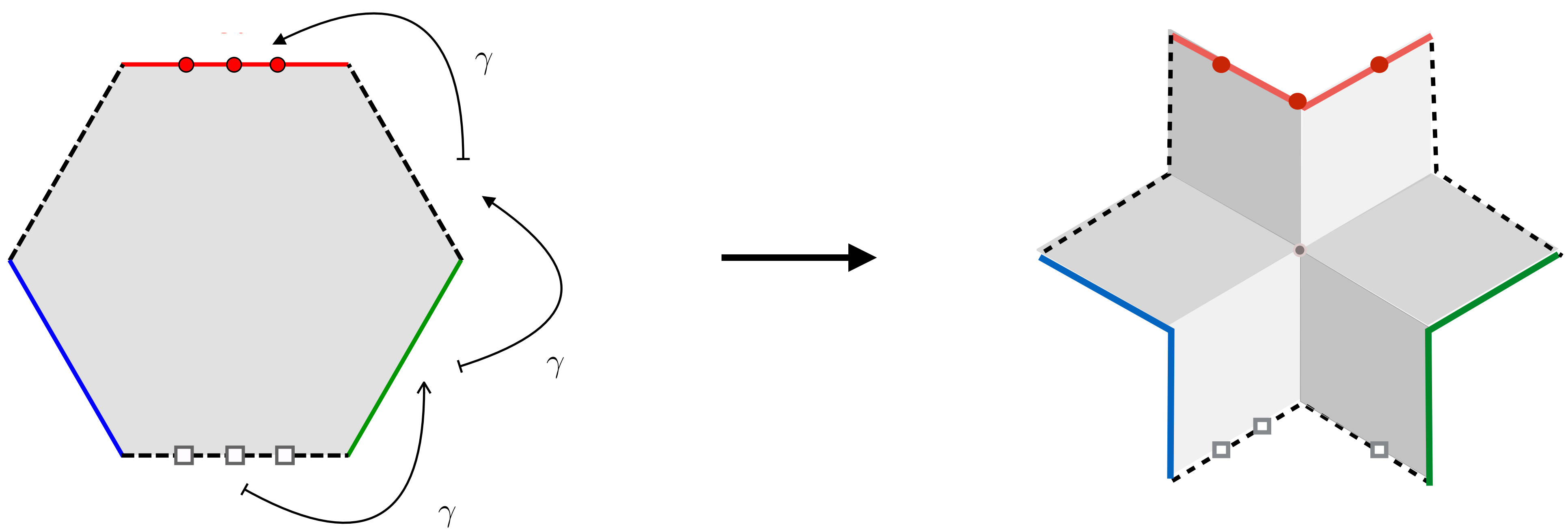
$$[\mathcal{C}_{123}^{\bullet\bullet\bullet}]^{\text{asympt}} = \sum_{\alpha_i \cup \bar{\alpha}_i = \mathbf{u}_i} (-1)^{|\alpha_1|+|\alpha_2|+|\alpha_3|} \, w_{\ell_{31}}(\alpha_1, \bar{\alpha}_1) \, w_{\ell_{12}}(\alpha_2, \bar{\alpha}_2) \, w_{\ell_{23}}(\alpha_3, \bar{\alpha}_3)$$
$$\times \quad \mathrm{H}(\alpha_1|\alpha_3|\alpha_2)\mathrm{H}(\bar{\alpha}_2|\bar{\alpha}_3|\bar{\alpha}_1) \; .$$

explicit Ansatz for the hexagon amplitudes $\mathrm{H}(\alpha_1|\alpha_3|\alpha_2)$ obeying form-factor-like axioms

Hexagon as a form factor

[Basso, Komatsu, Vieira, 15]

the hexagon can be seen as the form factor of a twist-like operator inducing a curvature excess of 180 degrees **[Cardy, Castro-Alvaredo, Doyon, 06]**



solution from bootstrap (form factor axioms)

$$\mathrm{H}^{\,A_1\dot{A}_1\cdots} = (-1)^{\mathfrak{f}} \prod_{i<j} h_{ij} \langle \chi_N^{\dot{A}_N} \ldots \chi_1^{\dot{A}_1} | \mathcal{S} | \chi_1^{A_1} \ldots \chi_N^{A_N} \rangle$$

dynamical part

matrix part

Computing the asymptotic part

the asymptotic part of the three point function can be written as a sum over partitions for the rapidities

$$[\mathcal{C}_{123}^{\bullet\circ\circ}]^{\text{asympt}} \equiv \mathscr{A} = \sum_{\alpha\cup\bar{\alpha}=\mathbf{u}} (-1)^{|\bar{\alpha}|} \prod_{j\in\alpha} e^{ip(u_j)\ell_R} \prod_{j\in\alpha,k\in\bar{\alpha}} \frac{1}{h(u_k,u_j)}$$

integral representation **[Kostov, 12; Bettelheim, Kostov, 14]**

$$\mathscr{A} = \sum_{n=0}^{M} \frac{1}{n!} \oint_{\mathcal{C}_{\mathbf{u}}} \prod_{j=1}^{n} \frac{dz_j}{2\pi\epsilon} F(z_j) \prod_{j<k}^{n} h(z_j,z_k)\, h(z_k,z_j)$$

fermionic dipole representation $\longrightarrow$ Fredholm determinant (Pfaffian)

$$h(u,v)h(v,u) = \langle : e^{\phi(u^+)-\phi(u^-)} :: e^{\phi(v^+)-\phi(v^-)} : \rangle$$

[Basso, Coronado, Komatsu, Lam, Zhong, 17]

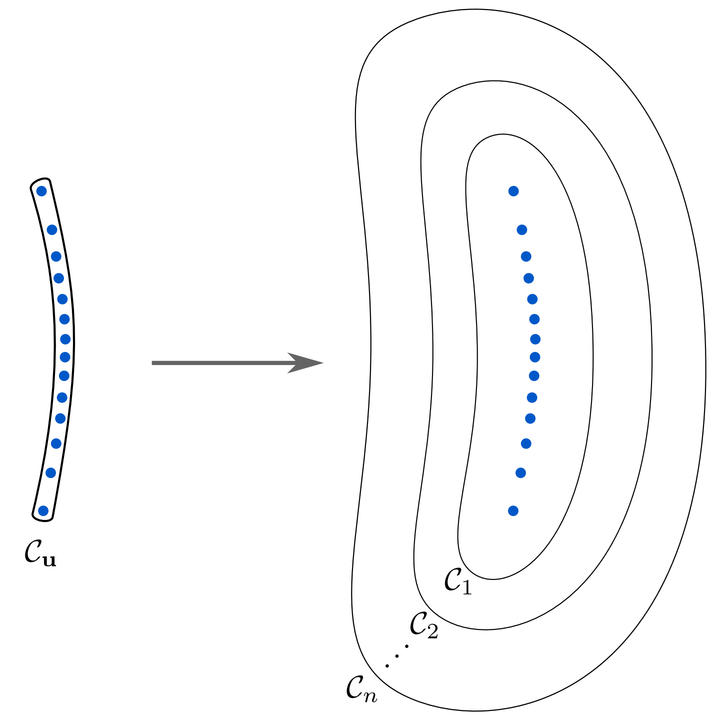
Clustering

$\epsilon \sim 1/L_1$

[Jiang, Komatsu, Kostov, DS, 16]

$$\mathscr{A} = \sum_{n=0}^{N} \frac{1}{n!} \oint_{\mathcal{C}_\mathbf{u}} \prod_{j=1}^{n} \frac{dz_j}{2\pi\epsilon} F(z_j) \prod_{j<k}^{n} \Delta(z_j, z_k)$$

manipulate the integrals by separating the contours



catch the poles of

$$\Delta(u,v) = 1 + \frac{i\epsilon/2}{u - v - i\epsilon} - \frac{i\epsilon/2}{u - v + i\epsilon}$$

and this gives rise to clustered integrals, *e.g.*

[Moore, Nekrasov, Shatashvili, 98]

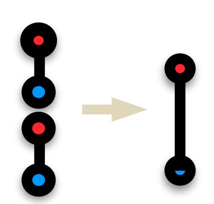
[Borodin, Corwin, 11]

[Bourgine, 14] [Menegelli, Yang, 14]

[Basso, Sever, Vieira, 13], …

$$\oint_{\mathcal{C}_\mathbf{u}} \oint_{\mathcal{C}_\mathbf{u}} \frac{dz_1 dz_2}{(2\pi\epsilon)^2} F(z_1)F(z_2)\Delta(z_1,z_2) = \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{dz_1 dz_2}{(2\pi\epsilon)^2} F(z_1)F(z_2)\Delta(z_1,z_2)$$
$$+ \oint_{\mathcal{C}_2} \frac{dz_2}{(2\pi\epsilon)} F(z_2)F(z_2+i\epsilon)$$

cluster of length 2

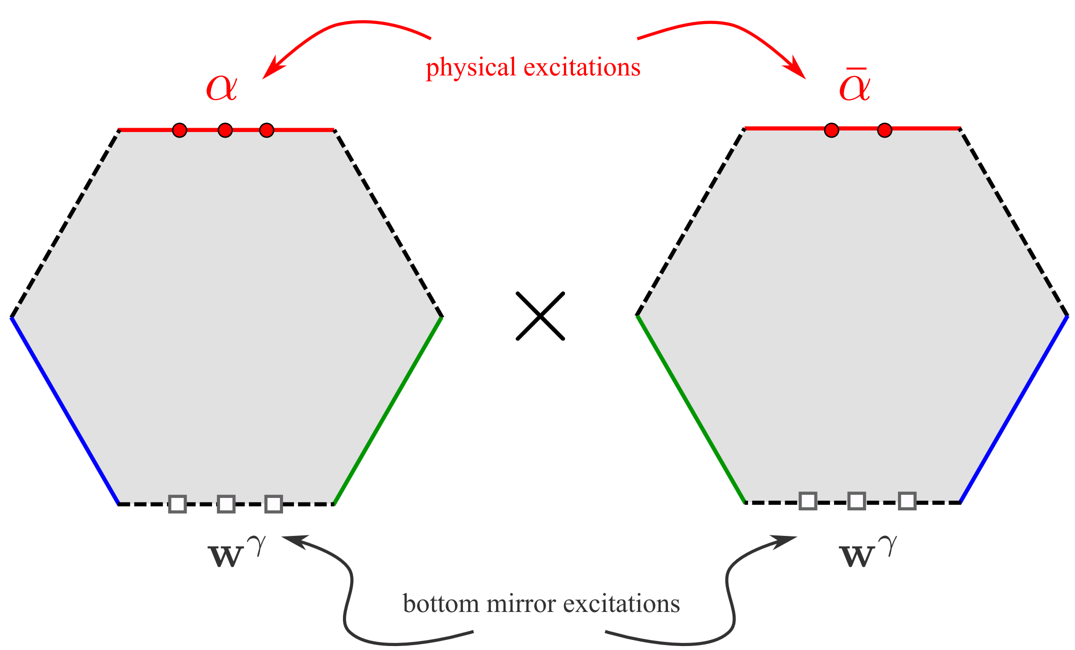


Mirror particle contribution

[Basso, Komatsu, Vieira, 15]

the asymptotic contribution is “dressed” by the mirror particles

(one non-BPS state and one mirror channel shown)



e.g. bottom channel: $$[\mathcal{C}^{\bullet\circ\circ}]^{\text{bottom}} = \int_{-\infty}^{\infty} d\mathbf{w}\, \mu(\mathbf{w}^{\gamma})\, e^{ip(\mathbf{w}^{\gamma})\ell_B}\, T(\mathbf{w}^{\gamma})\ h^{\neq}(\mathbf{w}^{\gamma}, \mathbf{w}^{\gamma})\ h(\mathbf{u}, \mathbf{w}^{-3\gamma})$$

sum over the rapidities **w** and polarisations of the mirror particles emitted by one hexagon and absorbed by the other (insertion of a resolution of identity)

mirror particles from different channels interact non-trivially

Structure of the three point function in the semiclassical limit

su(2) and sl(2) sectors

$su(2)_L \oplus su(2)_R \simeq so(4) \in so(6)$

I II

[Jiang, Komatsu, Kostov, D.S., 16]

- all-loop asymptotic result for type I-I-II correlators

$$\log[\mathcal{C}_{123}^{\bullet\bullet\bullet}]^{\text{asympt}}_{\mathfrak{su}(2)} = -\frac{1}{\epsilon}\oint_{\mathcal{C}_{\mathbf{u}_1\cup\mathbf{u}_2}}\frac{du}{2\pi}\,\mathrm{Li}_2\left[e^{i\tilde{p}_L^{(1)}+i\tilde{p}_L^{(2)}-i\tilde{p}_R^{(3)}}\right]-\frac{1}{\epsilon}\oint_{\mathcal{C}_{\mathbf{u}_3}}\frac{du}{2\pi}\,\mathrm{Li}_2\left[e^{i\tilde{p}_R^{(3)}+i\tilde{p}_L^{(2)}-i\tilde{p}_L^{(1)}}\right],$$

$$\log[\mathcal{C}_{123}^{\bullet\bullet\bullet}]^{\text{asympt}}_{\mathfrak{sl}(2)} = \frac{1}{\epsilon}\oint_{\mathcal{C}_{\mathbf{u}_1\cup\mathbf{u}_2}}\frac{du}{2\pi}\,\mathrm{Li}_2\left[e^{i\hat{p}_L^{(1)}+i\hat{p}_L^{(2)}-i\hat{p}_R^{(3)}}\right]+\frac{1}{\epsilon}\oint_{\mathcal{C}_{\mathbf{u}_3}}\frac{du}{2\pi}\,\mathrm{Li}_2\left[e^{i\hat{p}_R^{(3)}+i\hat{p}_L^{(2)}-i\hat{p}_L^{(1)}}\right].$$

- mirror contribution in the bottom channel at strong coupling (one non-BPS operator)

$$\log[\mathcal{C}_{123}^{\bullet\circ\circ}]^{\text{bottom}}_{su(2)} = \log[\mathcal{C}_{123}^{\bullet\circ\circ}]^{\text{bottom}}_{sl(2)} = \frac{1}{\epsilon}\oint_U\frac{du}{2\pi}\left(\mathrm{Li}_2\left[e^{i(\widehat{p}^{(2)}+\widehat{p}^{(3)}-\widehat{p}^{(1)})}\right]-\mathrm{Li}_2\left[e^{i(\widetilde{p}^{(2)}+\widetilde{p}^{(3)}-\widetilde{p}^{(1)}(x))}\right]\right)$$

AdS part

S part

matches contributions from strong coupling computations via string sigma model

[Kazama, Komatsu, 13; Kazama, Komatsu, Nishimura, 16]

Structure of the three point function in the semiclassical limit

Full mirror contribution in all channels

[Kazama, Komatsu, 13; Kazama, Komatsu, Nishimura, 16]

$$\log[\mathcal{C}^{\bullet\bullet\bullet}_{123}]^{\text{wrapping}}_{su(2)} = \sum_{\{i,j,k\}=\{1,2,3\}} \frac{1}{\epsilon} \oint_U \frac{du}{2\pi} \left(\mathrm{Li}_2 \left[e^{i(\widehat{p}^{(i)}+\widehat{p}^{(j)}-\widehat{p}^{(k)})} \right] - \mathrm{Li}_2 \left[e^{i(\widetilde{p}^{(i)}+\widetilde{p}^{(j)}-\widetilde{p}^{(k)})} \right] \right)$$

$$+ \frac{1}{\epsilon} \oint_U \frac{du}{2\pi} \left(\mathrm{Li}_2 \left[e^{i(\widehat{p}^{(3)}+\widehat{p}^{(1)}+\widehat{p}^{(2)})} \right] - \mathrm{Li}_2 \left[e^{i(\widetilde{p}^{(3)}+\widetilde{p}^{(1)}+\widetilde{p}^{(2)})} \right] \right)$$

$$- \sum_{j=1}^{3} \frac{1}{\epsilon} \oint_U \frac{du}{2\pi} \left(\mathrm{Li}_2 \left[e^{2i\widehat{p}^{(j)}} \right] - \mathrm{Li}_2 \left[e^{2i\widetilde{p}^{(j)}} \right] \right)$$

Analogy with Liouville:

$$C_{\alpha_1,\alpha_2,\alpha_3} = \frac{\left[b^{\frac{2}{b}-2b} \mu \right]^{Q-\alpha_1-\alpha_2-\alpha_3} \Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}$$

Mirror excitations at strong coupling

$$\epsilon = \frac{1}{2g}$$

$$[\mathcal{C}^{\bullet\circ\circ}]^{\rm bottom} = \sum_{\vec{n}} \frac{\mathrm{B}[\vec{n}]}{\prod_a n_a!}$$

$$\mathrm{B}[\vec{n}] = (-1)^n \int_{-\infty}^{\infty} \prod_a \prod_{j=1}^{n_a} \frac{dz_j^a}{2\pi\epsilon}\, \mu_a^\gamma(z_j^a)\, \mathrm{T}_a^\gamma(z_j^a) \times \prod_{\substack{a \\ 1\le i<j\le n_a}} H_{aa}^\gamma(z_i^a, z_j^a) \prod_{\substack{a<b \\ 1\le i\le n_a \\ 1\le j\le n_b}} H_{ab}^\gamma(z_i^a, z_j^b)$$

multiple integrals coupling bound states with different lengths $\quad H_{ab}^\gamma(u,v) \equiv h_{ab}(u^\gamma, v^\gamma)\, h_{ba}(v^\gamma, u^\gamma)$

strong coupling limit: $\quad H_{ab}^\gamma(u,v) \simeq \frac{u-v-i\epsilon\frac{a-b}{2}}{u-v-i\epsilon\frac{a+b}{2}}\;\frac{u-v+i\epsilon\frac{a-b}{2}}{u-v+i\epsilon\frac{a+b}{2}} = 1 - \delta_{ab}(u-v)$

transfer matrix for the bound state *a* $\quad \mathrm{T}_a(u) \to \widetilde{g}^a(u)\left[(a+1) - a\,f - a\,\bar{f} + (a-1) f\bar{f}\right]$

$$f(u) = e^{i\mathcal{G}(x)}\,, \qquad \bar{f}(u) = e^{-i\mathcal{G}(1/x)}\,, \qquad \mathcal{G}(x) = \frac{1}{i}\sum_j \ln\frac{x-x_j^-}{x-x_j^+} \;\to\; \epsilon \sum_j \frac{x'(u_j)}{x-x(u_j)}$$

taking into account the poles in the measure pinching the contour of integration

⟶ clustering of bound states

Clustering of mirror bound states

The generating functional of transfer matrices at strong coupling **[Beisert, 06]** **[Kazakov, Sorin, Zabrodin, 07]**

$$\mathrm{Sdet}(1-z\,\mathrm{G})^{-1}=\frac{(1-zy_1)(1-zy_2)}{(1-zx_1)(1-zx_2)}=\sum_a z^a\,\mathrm{T}_a$$

$$\mathrm{Str}(1-z\,\mathrm{G})^{-1}=z\frac{\mathrm{d}}{\mathrm{d}z}\log\mathrm{Sdet}(1-z\,\mathrm{G})^{-1}=\sum_a z^a\,\mathrm{t}_a$$

$G=\mathrm{diag}(x_1,x_2|y_1,y_2)$

SU(2|2) monodromy matrix

$$\frac{\mathrm{t}_n}{n}=\sum_{\vec{n}\,:\,\sum_a n_a\,a=n}(-1)^{k-1}(k-1)!\prod_a\frac{\mathrm{T}_a^{n_a}}{n_a!}$$

$\mathrm{t}_n=\mathrm{Tr}\;\mathbf{T}^n$

a set of bound states $\vec{n}=\{n_1,n_2,\cdots\}$ clusters into

$$\{q_1,\cdots,q_m\}=\{\underbrace{1\cdots1}_{d_1},\cdots,\underbrace{l,\cdots,l}_{d_l},\cdots\}\mapsto\vec{d}=\{d_1,d_2,\cdots\}.$$

$$\mathrm{B}_{\vec{d}}=\prod_l\frac{1}{d_l!}\times\prod_{j=1}^m\int\frac{dz_j}{2\pi\epsilon}\frac{\mathrm{t}_{q_j}(z_j)}{q_j^2}$$

$$[\mathcal{C}^{\bullet\circ\circ}]^{\mathrm{bottom}}=\sum_{\vec{d}}\mathrm{B}_{\vec{d}}=\exp\int\frac{dz}{2\pi\epsilon}\sum_n\frac{\mathrm{t}_n(z)}{n^2}$$

Clustering in cross channels

gluing the hexagons on two sides $\longrightarrow$ divergence in the original BKV proposal

$$\sum_{n_R=0}^{\infty} \frac{1}{n_R!} \sum_{n_L=0}^{\infty} \frac{1}{n_L!} \int_{\mathbb{R}} d\mu(\mathbf{w}_L^{\gamma}) \int_{\mathbb{R}} d\mu(\mathbf{w}_R^{\gamma}) \frac{1}{h(\mathbf{w}_L^{\gamma}, \mathbf{w}_R^{\gamma}) h(\mathbf{w}_R^{\gamma}, \mathbf{w}_L^{\gamma})}$$

double poles

$$\times \frac{e^{-E(\mathbf{w}_L)\ell_L}}{h(\mathbf{w}_L^{-\gamma}, \mathbf{u})} T(\mathbf{w}_L^{-\gamma}) h^{\neq}(\mathbf{w}_L^{\gamma}, \mathbf{w}_L^{\gamma})$$

left channel L

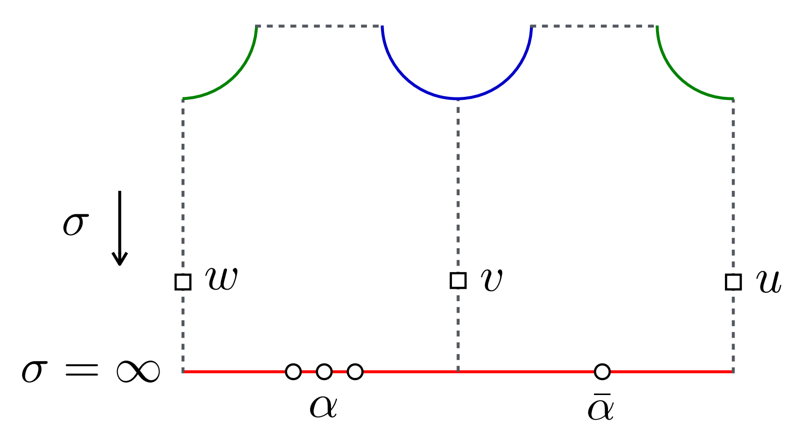
$$\times \frac{e^{-E(\mathbf{w}_R)\ell_R}}{h(\mathbf{w}_R^{-5\gamma}, \mathbf{u})} T(\mathbf{w}_R^{-\gamma}) h^{\neq}(\mathbf{w}_R^{\gamma}, \mathbf{w}_R^{\gamma})$$

right channel R

$$\times \sum_n \frac{(-1)^n}{n!} \oint_{\mathcal{C}_{\mathbf{u}}} d\mu(\mathbf{z})\, e^{ip(\mathbf{z})\ell_R} \frac{h(\mathbf{z}, \mathbf{w}_L^{-\gamma}) h(\mathbf{w}_L^{-\gamma}, \mathbf{z})}{h(\mathbf{z}, \mathbf{w}_R^{-\gamma}) h(\mathbf{w}_R^{-\gamma}, \mathbf{z})} \frac{h^{\neq}(\mathbf{z}, \mathbf{z})}{h(\mathbf{u}, \mathbf{z})}$$

"gluing more slowly"

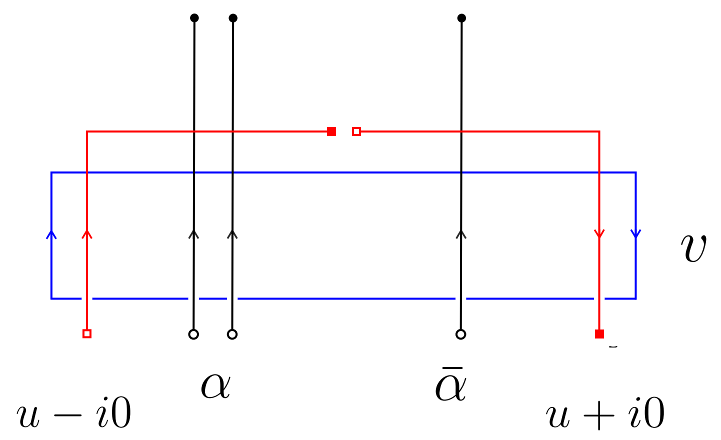
[Basso, Gonçalves, Komatsu, 17]



shift the poles out of the real axis / introduce a volume regulator: $\sigma \leq R$

Clustering in cross channels

"gluing more slowly"



$$\frac{T_a(u)\, T_a(v)}{h_{aa}(u,v) h_{aa}(v,u)} \quad \longrightarrow \quad \frac{\mathrm{Tr}_{\,12}\, S_{aa}^{-1}(u-v+i0)\; S_{aa}(u-v-i0)\; \mathbf{T}_a(u)\; \mathbf{T}_a(v)}{h_{aa}(u+i0,v) h_{aa}(v,u-i0)}$$

$$\frac{1}{h_{aa}(u+i0,v)\; h_{aa}(v,u-i0)} \simeq \frac{(u-v)^2+a^2}{(u-v+i0)(u-v-i0)} \simeq 1 + \infty\; \delta(u-v)$$

shift the poles out of the real axis / introduce a volume regulator: $\sigma \le R/2$

$$\int \frac{du}{2\pi} \frac{1}{(u-v+i0)(u-v-i0)}$$

$$= \int_{-\infty}^{\infty} \frac{p'(u) dp(u)}{2\pi} \int_0^{R/2} d\sigma \int_0^{R/2} d\sigma' e^{i(p(u)-p(v)+i0)\sigma} e^{i(p(v)-p(u)+i0)\sigma'}$$

$$= \int_0^{R/2} p'(u) d\sigma = R\, p'(u)/2$$

Clustering in cross channels

two kinds of contributions: **[Basso, Gonçalves, Komatsu, 17]**

- term diverging with the volume R which will be compensated by the normalisation
- the divergent term is the TBA correction to the energy (non-trivial check!)

strong coupling:

$$\exp \frac{R}{2} \delta E_{\mathbf{u}}^{\mathrm{mirror}}$$

[Gromov, 09]

$$\delta E_{\mathbf{u}}^{\mathrm{mirror}} = \int dz\, p'(z) \sum_{n=1}^{\infty} \frac{\mathrm{t}_n(z)\mathrm{t}_n(z)}{n}$$

[Kostov, D.S., in progress]

- finite terms, *e.g.* from the interaction of two bound states of size *a*

[BGK, 17] check to fourth loop order against perturbative computation

strong coupling:

$$\mathrm{Tr}_{12}\ \ \mathbf{T}_a(u)\ \mathbf{T}_a(u)\ \partial_v \log S_{aa}(v)|_{v=0} \quad \to \quad \frac{t_{2a}}{a^2} + \text{lower}$$

[Kostov, D.S., in progress]

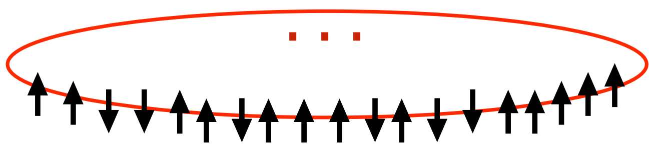
give rise to the dilogarithm

presumably cancelled in the combinatorics

Conclusion and outlook

- three point correlation functions can be obtained from hexagon decomposition and resummation of virtual particles
- the procedure should be very general and applicable for any integrable field theory (*e.g.* massive boson **[Bajnok, Janik, 15-17]**, 2d CFT, etc)
- higher point functions also obtainable by tessellation with hexagons **[Komatsu, Fleury, 16; Eden, Sfondrini, 16]**
- computational techniques for the virtual particle contribution is important for evaluating loop integrals in the perturbative gauge theory **[Basso, Dixon, 17]**
- importance of the curvature defect operators introduced in the '70s and used in various contexts; e.g. **[Castro-Alvaredo, Doyon, Fioravanti, 17]**

Happy birthday!



Happy birthday!