# Scalar products in the algebraic Bethe ansatz

N.A. Slavnov

Steklov Mathematical Institute, Moscow

## Correlation functions in quantum integrable systems and beyond

On the occasion of the 60th birthday of JEAN–MICHEL MAILLET

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# Scalar products in the algebraic Bethe ansatz

N.A. Slavnov

Steklov Mathematical Institute, Moscow

in collaboration with

A. Hutsalyuk, A. Liashyk, S. Pakuliak, and E. Ragoucy

*Scalar products of Bethe vectors in the models with* $\mathfrak{gl}(m|n)$ *symmetry*

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## Preliminary remarks

Within the framework of the algebraic Bethe ansatz, calculating the scalar products of Bethe vectors is a necessary step in the problem of correlation functions.

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Within the framework of the algebraic Bethe ansatz, calculating the scalar products of Bethe vectors is a necessary step in the problem of correlation functions.

For $\mathfrak{gl}(2)$ based models, the scalar products can be calculated by direct methods. However, in the case of the models with higher rank symmetry, application of the direct methods leads to serious technical difficulties. One should find some other ways of calculating the scalar products.

## Preliminary remarks

For simplicity we consider Bethe ansatz solvable models described by $Y(\mathfrak{gl}(n))$. This means that the $R$-matrix is fixed as

$$R(u,v)=\mathbf{I}+g(u,v)\mathbf{P}, \qquad g(u,v)=\frac{c}{u-v}, \quad c \text{ is a constant}$$

However, the main idea of a new method perfectly works for the models with $Y(\mathfrak{gl}(m|n))$ symmetries and their $q$-deformation.

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However, the main idea of a new method perfectly works for the models with $Y(\mathfrak{gl}(m|n))$ symmetries and their $q$-deformation.

The advantage of the new method to calculate the scalar products of the Bethe vectors is that this approach is rank independent.

The main tool of the new method is a *coproduct formula for the Bethe vectors.*

- Introduction, notation
  - Bethe vectors
    - Formulation of the problem
      - Composite model
        - Composite model and scalar products
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# Introduction

We begin with the $RTT$-relation

$$R(u,v)\Big(T(u)\otimes I\Big)\Big(I\otimes T(v)\Big)=\Big(I\otimes T(v)\Big)\Big(T(u)\otimes I\Big)R(u,v)$$

with an $n\times n$ monodromy matrix $T(u)$.

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$RTT$-relation implies the following commutation relations

$$[T_{i,j}(u),T_{k,l}(v)]=g(u,v)\Big(T_{k,j}(v)T_{i,l}(u)-T_{k,j}(u)T_{i,l}(v)\Big)$$

## Introduction

We assume that $T_{i,j}(u)$ act in some Hilbert space $\mathcal{H}$ and a dual space $\mathcal{H}^*$ with respectively pseudovacuum $|0\rangle \in \mathcal{H}$ and dual pseudovacuum $\langle 0| \in \mathcal{H}^*$ vectors.

$$T_{i,i}(u)|0\rangle = \lambda_i(u)|0\rangle$$

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where $\lambda_i(u)$ are some functions. Similarly,

$$\langle 0|T_{i,i}(u) = \lambda_i(u)\langle 0|$$

$$\langle 0|T_{i,j}(u) = 0, \qquad i < j$$

Below we deal with ratios $\quad \alpha_i(u) = \dfrac{\lambda_i(u)}{\lambda_{i+1}(u)}$

# Notation

Rational functions used below

$$g(u,v) = \frac{c}{u-v}$$

$$f(u,v) = 1 + g(u,v) = \frac{u-v+c}{u-v}$$

## Notation

Sets of variables

$$\bar{u} = \{u_1, \dots, u_a\}, \qquad \bar{v} = \{v_1, \dots, v_b\}, \qquad \bar{t}^i = \{t^i_1, \dots, t^i_{r_i}\}$$

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Products over the sets

$$\lambda_i(\bar{u}) = \prod_{u_k \in \bar{u}} \lambda_i(u_k)$$

$$\alpha_i(\bar{t}^i) = \prod_{t^i_k \in \bar{t}^i} \alpha_i(t^i_k)$$

$$T_{i,j}(\bar{u}) = \prod_{u_k \in \bar{u}} T_{i,j}(u_k)$$

$$g(\bar{u}, v_j) = \prod_{u_k \in \bar{u}} g(u_k, v_j)$$

$$f(\bar{u}, \bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j, v_k)$$

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Generically, a Bethe vector $\mathbb{B}(\bar{u})$ depends on a set of complex variables $\bar{u} = \{u_1, \dots, u_N\}$ called Bethe parameters. If these parameters satisfy a set of equations (Bethe equations), then the corresponding vector is an eigenvector of the transfer matrix $\mathrm{tr}\, T(z)$. In this case we call it *on-shell* Bethe vector. Otherwise, if $\bar{u}$ are generic complex numbers, then we deal with a generic Bethe vector.

## Bethe vectors

Bethe vectors in $\mathfrak{gl}(2)$ based models.

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

There exists only one creation operator $B(u)$. Bethe vectors have the following form:

$$\mathbb{B}(\bar{u}) = B(\bar{u})|0\rangle$$

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$$\mathbb{B}(\bar{u}) = B(\bar{u})|0\rangle \equiv B(u_1)\dots B(u_N)|0\rangle$$

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For $\mathfrak{gl}(n)$ based models with $n > 2$ Bethe vectors are special polynomials in $T_{i,j}$ applied to $|0\rangle$.

## Generic Bethe vector for $\mathfrak{gl}(3)$ based models

$$\mathbb{B}(\bar{u};\bar{v}) = \sum Z(\bar{v}_{\rm I}|\bar{u}_{\rm I})\frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II})f(\bar{v}_{\rm II},\bar{v}_{\rm I})}{\lambda_2(\bar{v}_{\rm II})\lambda_2(\bar{u})f(\bar{v},\bar{u})}T_{1,3}(\bar{u}_{\rm I})T_{1,2}(\bar{u}_{\rm II})T_{2,3}(\bar{v}_{\rm II})|0\rangle$$

The sum is taken over partitions $\bar{u} \Rightarrow \bar{u}_{\rm I} \cup \bar{u}_{\rm II}$ and $\bar{v} \Rightarrow \bar{v}_{\rm I} \cup \bar{v}_{\rm II}$ such that $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I}$. $Z(\bar{x}|\bar{y})$ is the partition function of the six-vertex model with domain wall boundary condition. Everywhere the shorthand notation for the products is used.

A Bethe vector $\mathbb{B}(\bar{u};\bar{v})$ depends on two sets of variables $\bar{u}$ and $\bar{v}$, such that $\#\bar{u} = r_1$ and $\#\bar{v} = r_2$, $r_i = 0, 1, \dots$. Generically we do not impose any restriction on the Bethe parameters.

## Bethe vectors for $\mathfrak{gl}(n)$ based models

In $\mathfrak{gl}(n)$ based models Bethe vectors depend on $n-1$ sets of parameters: $\bar t=\{\bar t^1,\bar t^2,\dots,\bar t^{n-1}\}$. In its turn, each set $\bar t^\mu$ consists of individual Bethe parameters: $\bar t^\mu=\{\bar t^\mu_1,\dots,\bar t^\mu_{r_\mu}\}$, where $r_\mu=\#\bar t^\mu$.

$$\mathbb{B}(\bar t)=\mathbb{B}(\{t^1_1,\dots,t^1_{r_1}\},\{t^2_1,\dots,t^2_{r_2}\}\dots,\{t^{n-1}_1,\dots,t^{n-1}_{r_{n-1}}\})$$

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Bethe vector can be presented in the form

$$\mathbb{B}(\bar t) = \mathcal{P}(T_{i,j})|0\rangle$$

where $\mathcal{P}(T_{i,j})$ is a polynomial in $T_{i,j}$. It is called *pre-Bethe vector* or *universal weight function.*

## Bethe vectors for $\mathfrak{gl}(n)$ based models

- Nested algebraic Bethe ansatz

**P. Kulish, N. Reshetikhin, '81, '83**

Bethe vectors of $\mathfrak{gl}(n)$ models are constructed recursively in terms of Bethe vectors $\mathfrak{gl}(n-1)$ based models.

- Other formulations of nested Bethe ansatz

**V. Tarasov, A. Varchenko '94, '96**

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These methods can give either recursions or even explicit formulas for Bethe vectors.

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One of the defining properties of Bethe vectors is the property of the universal weight function with respect to coproduct.

## Dual Bethe vectors

Dual Bethe vectors can be obtained by transposition $T_{i,j} \to T_{j,i}$ and $|0\rangle \to \langle 0|$

$$\mathbb{C}(\bar{u}) = \langle 0|C(\bar{u})$$

$$\mathbb{C}(\bar{u};\bar{v}) = \sum Z(\bar{v}_{\rm I}|\bar{u}_{\rm I})\frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II})f(\bar{v}_{\rm II},\bar{v}_{\rm I})}{\lambda_2(\bar{v}_{\rm II})\lambda_2(\bar{u})f(\bar{v},\bar{u})}\langle 0|T_{3,2}(\bar{v}_{\rm II})T_{2,1}(\bar{u}_{\rm II})T_{3,1}(\bar{u}_{\rm I})$$

Generically, dual Bethe vectors depend on $n-1$ sets of parameters: $\bar{s} = \{\bar{s}^1, \bar{s}^2, \dots, \bar{s}^{n-1}\}$. In its turn, each set $\bar{s}^\mu$ consists of individual Bethe parameters: $\bar{s}^\mu = \{\bar{s}^\mu_1, \dots, \bar{s}^\mu_{r_\mu}\}$, where $r_\mu = \#\bar{s}^\mu$.

$$\mathbb{C}(\bar{s}) = \mathbb{C}(\{s^1_1, \dots, s^1_{r_1}\}, \{s^2_1, \dots, s^2_{r_2}\} \dots, \{s^{n-1}_1, \dots, s^{n-1}_{r_{n-1}}\})$$

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## Scalar product of Bethe vectors

$$S(\bar{s}|\bar{t}) = \mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \langle 0|\mathcal{P}^{T}(\bar{s})\mathcal{P}(\bar{t})|0\rangle$$

Calculation of the scalar products is the problem of normal ordering.

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$$S(\bar{s}|\bar{t}) = \sum Z(\bar{s}^2_{\rm I}|\bar{s}^1_{\rm I})Z(\bar{t}^2_{\rm I}|\bar{t}^1_{\rm I})\frac{f(\bar{s}^1_{\rm I},\bar{s}^1_{\rm II})f(\bar{s}^2_{\rm II},\bar{s}^2_{\rm I})f(\bar{t}^1_{\rm I},\bar{t}^1_{\rm II})f(\bar{t}^2_{\rm II},\bar{t}^2_{\rm I})}{\lambda_2(\bar{s}^2_{\rm II})\lambda_2(\bar{s}^1)\lambda_2(\bar{t}^2_{\rm II})\lambda_2(\bar{t}^1)f(\bar{s}^2,\bar{s}^1)f(\bar{t}^2,\bar{t}^1)}$$

$$\times\langle 0|T_{3,2}(\bar{s}^2_{\rm II})T_{2,1}(\bar{s}^1_{\rm II})T_{3,1}(\bar{s}^1_{\rm I})T_{1,3}(\bar{t}^1_{\rm I})T_{1,2}(\bar{t}^1_{\rm II})T_{2,3}(\bar{t}^2_{\rm II})|0\rangle$$

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$\mathfrak{gl}(2)$ case.

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$$\frac{\lambda_1(u)}{\lambda_2(u)} = \alpha_1(u) \equiv \alpha(u)$$

## Scalar product as a sum over partitions

$\mathfrak{gl}(2)$

$$\mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \sum \alpha(\bar{v}_{\rm I})\alpha(\bar{u}_{\rm II})W_{\textsf{part}}\begin{pmatrix}\bar{v}_{\rm I} & \bar{u}_{\rm I}\\ \bar{v}_{\rm II} & \bar{u}_{\rm II}\end{pmatrix}$$

The sum is taken over partitions $\bar{u} \Rightarrow \bar{u}_{\rm I} \cup \bar{u}_{\rm II}$ and $\bar{v} \Rightarrow \bar{v}_{\rm I} \cup \bar{v}_{\rm II}$ such that $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I}$. The coefficients $W_{\textsf{part}}$ are rational functions of the Bethe parameters. They are model independent.

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The sum is taken over partitions $\bar{u} \Rightarrow \bar{u}_{\rm I} \cup \bar{u}_{\rm II}$ and $\bar{v} \Rightarrow \bar{v}_{\rm I} \cup \bar{v}_{\rm II}$ such that $\#\bar{u}_{\rm I} = \#\bar{v}_{\rm I}$. The coefficients $W_{\sf part}$ are rational functions of the Bethe parameters. They are model independent.

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Example:

$$\frac{\langle 0|C(v)B(u)|0\rangle}{\lambda_2(v)\lambda_2(u)} = g(v,u)\frac{\langle 0|\Big(A(u)D(v) - A(v)D(u)\Big)|0\rangle}{\lambda_2(v)\lambda_2(u)}$$

$$= g(v,u)\frac{\lambda_1(u)\lambda_2(v) - \lambda_1(v)\lambda_2(u)}{\lambda_2(v)\lambda_2(u)} = \alpha(u)g(v,u) + \alpha(v)g(u,v)$$

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A particular case of $W_{\sf part}$ is called *Highest coefficient*:

$$Z(\bar{v}|\bar{u}) = W_{\sf part}\begin{pmatrix}\bar{v} & \bar{u}\\ \emptyset & \emptyset\end{pmatrix}$$

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$$\mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \alpha(\bar{v}) Z(\bar{v}|\bar{u}), \qquad \text{if} \qquad \alpha(u_k) = 0, \quad \forall k$$

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A particular case of $W_{\textsf{part}}$ is called *Highest coefficient*.

For $\mathfrak{gl}(2)$ based models, the highest coefficient is equal to the partition function of the six-vertex model with domain wall boundary condition.

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Conjugated Highest coefficient is defined as

$$\overline{Z}(\bar v|\bar u)=W_{\sf part}\begin{pmatrix}\emptyset&\emptyset\\ \bar v&\bar u\end{pmatrix}$$

It is easy to show that

$$\overline{Z}(\bar v|\bar u)=Z(\bar u|\bar v)$$

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It is known that for $\mathfrak{gl}(2)$ based models a generic $W_{\textsf{part}}$ is proportional to the product of two highest coefficients.

**How to find explicitly this expression?**

## Scalar product as a sum over partitions

$\mathfrak{gl}(n)$

$$\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \sum \left( \prod_{k=1}^{n-1} \alpha_k(\bar{s}_{\rm I}^k)\alpha_k(\bar{t}_{\rm II}^k) \right) W_{\sf part} \begin{pmatrix} \bar{s}_{\rm I} & \bar{t}_{\rm I} \\ \bar{s}_{\rm II} & \bar{t}_{\rm II} \end{pmatrix}$$

The sum is taken over partitions $\bar{t}^k \Rightarrow \bar{t}_{\rm I}^k \cup \bar{t}_{\rm II}^k$ and $\bar{s}^k \Rightarrow \bar{s}_{\rm I}^k \cup \bar{s}_{\rm II}^k$ such that $\#\bar{t}_{\rm I}^k = \#\bar{s}_{\rm I}^k$, $k = 1, \dots, n-1$. The coefficients $W_{\sf part}$ are rational functions of the Bethe parameters.

They are model independent.

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They are model independent.

We can define the highest coefficient and its conjugated as

$$Z(\bar{s}|\bar{t}) = W_{\rm part}\begin{pmatrix} \bar{s} & \bar{t} \\ \emptyset & \emptyset \end{pmatrix}, \qquad \overline{Z}(\bar{s}|\bar{t}) = W_{\rm part}\begin{pmatrix} \emptyset & \emptyset \\ \bar{s} & \bar{t} \end{pmatrix} = Z(\bar{t}|\bar{s})$$

## Scalar product as a sum over partitions

$\mathfrak{gl}(n)$

$$\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \sum \left(\prod_{k=1}^{n-1} \alpha_k(\bar{s}^k_{\mathrm{I}})\alpha_k(\bar{t}^k_{\mathrm{II}})\right) W_{\mathsf{part}}\begin{pmatrix} \bar{s}_{\mathrm{I}} & \bar{t}_{\mathrm{I}} \\ \bar{s}_{\mathrm{II}} & \bar{t}_{\mathrm{II}} \end{pmatrix}$$

The sum is taken over partitions $\bar{t}^k \Rightarrow \bar{t}^k_{\mathrm{I}} \cup \bar{t}^k_{\mathrm{II}}$ and $\bar{s}^k \Rightarrow \bar{s}^k_{\mathrm{I}} \cup \bar{s}^k_{\mathrm{II}}$ such that $\#\bar{t}^k_{\mathrm{I}} = \#\bar{s}^k_{\mathrm{I}}$, $k = 1, \dots, n-1$. The coefficients $W_{\mathsf{part}}$ are rational functions of the Bethe parameters.
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**How to find generic $W_{\mathsf{part}}$ in terms of the highest coefficients?**

- Introduction, notation
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## Composite model

(Two-site model) **A. Izergin, V. Korepin '84**

$$T(u) = \underbrace{L_N(u)\ldots L_{m+1}(u)}_{T^{(2)}(u)} \cdot \underbrace{L_m(u)\ldots L_1(u)}_{T^{(1)}(u)}$$

$$T(u) = T^{(2)}(u)T^{(1)}(u)$$

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We call $T^{(\ell)}(u)$ *partial monodromy matrices.*
The matrix $T(u)$ is called *total monodromy matrix.*

We assume that the entries of the partial monodromy matrices $T^{(\ell)}_{ij}(u)$ act in the spaces $V^{(\ell)}$, while the entries of the total monodromy matrix $T_{i,j}(u)$ act in the tensor product $V^{(1)} \otimes V^{(2)}$.

## Composite model

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$$T(u) = T^{(2)}(u) T^{(1)}(u)$$

Commutation relations

$$R_{12}(u,v) T_1^{(\ell)}(u) T_2^{(\ell)}(v) = T_2^{(\ell)}(v) T_1^{(\ell)}(u) R_{12}(u,v)$$

$$[T_{i,j}^{(1)}(u), T_{k,l}^{(2)}(v)] = 0$$

# Composite model

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$$T(u) = \underbrace{L_N(u)\ldots L_{m+1}(u)}_{T^{(2)}(u)} \cdot \underbrace{L_m(u)\ldots L_1(u)}_{T^{(1)}(u)}$$

$$T(u) = T^{(2)}(u)T^{(1)}(u)$$

Action on $|0\rangle = |0\rangle^{(1)} \otimes |0\rangle^{(2)}$ and $\langle 0| = \langle 0|^{(1)} \otimes \langle 0|^{(2)}$

$$T^{(\ell)}_{j,j}(u)|0\rangle^{(\ell)} = \lambda^{(\ell)}_j(u)|0\rangle^{(\ell)}, \qquad T^{(\ell)}_{j,k}(u)|0\rangle^{(\ell)} = 0, \quad j > k$$

$$\langle 0|^{(\ell)}T^{(\ell)}_{j,j}(u) = \lambda^{(\ell)}_j(u)\langle 0|^{(\ell)}, \qquad \langle 0|^{(\ell)}T^{(\ell)}_{j,k}(u) = 0, \quad j < k$$

We set $\alpha^{(\ell)}_j(u) = \lambda^{(\ell)}_j(u)/\lambda^{(\ell)}_{j+1}(u)$.

## Composite model

(Two-site model) **A. Izergin, V. Korepin '84**

$$T(u) = \underbrace{L_N(u)\dots L_{m+1}(u)}_{T^{(2)}(u)} \cdot \underbrace{L_m(u)\dots L_1(u)}_{T^{(1)}(u)}$$

$$T(u) = T^{(2)}(u)T^{(1)}(u)$$

Partial Bethe vectors $\mathbb{B}^{(\ell)}(\bar{u})$ can be obtained from the total Bethe vectors $\mathbb{B}(\bar{u})$ via the replacements $T_{i,j} \to T_{i,j}^{(\ell)}$, $\lambda_i \to \lambda_i^{(\ell)}$, and $|0\rangle \to |0\rangle^{(\ell)}$. For instance,

$\mathfrak{gl}(2)$:
$$\mathbb{B}^{(\ell)}(\bar{u}) = \frac{T_{1,2}^{(\ell)}(\bar{u})}{\lambda_2^{(\ell)}(\bar{u})}|0\rangle^{(\ell)}$$

## Composite model

How to express the total Bethe vector in terms of the partial ones?

$$T(u) = T^{(2)}(u)T^{(1)}(u) \qquad \longrightarrow \qquad T_{i,j}(u) = T^{(2)}_{i,k}(u)T^{(1)}_{k,j}(u)$$

## Composite model

How to express the total Bethe vector in terms of the partial ones?

$$T(u) = T^{(2)}(u)T^{(1)}(u) \qquad \longrightarrow \qquad T_{i,j}(u) = T^{(2)}_{i,k}(u)T^{(1)}_{k,j}(u)$$

Thus, in $\mathfrak{gl}(2)$ we have: $B(u) = A^{(2)}(u)B^{(1)}(u) + B^{(2)}(u)D^{(1)}(u)$

$$\frac{B(\bar u)}{\lambda_2(\bar u)}|0\rangle = \prod_{k=1}^{n}\frac{1}{\lambda_2(u_k)}\Big(A^{(2)}(u_k)B^{(1)}(u_k) + B^{(2)}(u_k)D^{(1)}(u_k)\Big)|0\rangle^{(1)}\otimes|0\rangle^{(2)}$$

$$\mathbb{B}(\bar u) = \sum_{\text{part}} \alpha^{(2)}(\bar u_{\rm I}) f(\bar u_{\rm II}, \bar u_{\rm I})\, \mathbb{B}^{(1)}(\bar u_{\rm I}) \otimes \mathbb{B}^{(2)}(\bar u_{\rm II})$$

**A. Izergin, V. Korepin '84**

The sum is taken over partitions $\bar u \Rightarrow \bar u_{\rm I} \cup \bar u_{\rm II}$.

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# Composite model and scalar products

$$\mathbb{B}(\bar{u}) = \sum \alpha^{(2)}(\bar{u}_{\mathsf{i}})\mathbb{B}^{(1)}(\bar{u}_{\mathsf{i}})\mathbb{B}^{(2)}(\bar{u}_{\mathsf{ii}}) \cdot f(\bar{u}_{\mathsf{ii}}, \bar{u}_{\mathsf{i}})$$

$$\mathbb{C}(\bar{v}) = \sum \alpha^{(1)}(\bar{v}_{\mathsf{ii}})\mathbb{C}^{(1)}(\bar{v}_{\mathsf{i}})\mathbb{C}^{(2)}(\bar{v}_{\mathsf{ii}}) \cdot f(\bar{v}_{\mathsf{i}}, \bar{v}_{\mathsf{ii}})$$

## Composite model and scalar products

$$\mathbb{B}(\bar{u}) = \sum \alpha^{(2)}(\bar{u}_{\rm i})\mathbb{B}^{(1)}(\bar{u}_{\rm i})\mathbb{B}^{(2)}(\bar{u}_{\rm ii}) \cdot f(\bar{u}_{\rm ii}, \bar{u}_{\rm i})$$

$$\mathbb{C}(\bar{v}) = \sum \alpha^{(1)}(\bar{v}_{\rm ii})\mathbb{C}^{(1)}(\bar{v}_{\rm i})\mathbb{C}^{(2)}(\bar{v}_{\rm ii}) \cdot f(\bar{v}_{\rm i}, \bar{v}_{\rm ii})$$

Let $\#\bar{v} = \#\bar{u} = N$. Let us fix some partition $\bar{v} \Rightarrow \bar{v}_{\rm I} \cup \bar{v}_{\rm II}$ and $\bar{u} \Rightarrow \bar{u}_{\rm I} \cup \bar{u}_{\rm II}$ such that $\#\bar{v}_{\rm I} = \#\bar{u}_{\rm I} = m$, $m = 0, 1, \ldots, N$.

## Composite model and scalar products

$$\mathbb{B}(\bar{u}) = \sum \alpha^{(2)}(\bar{u}_{\rm i})\mathbb{B}^{(1)}(\bar{u}_{\rm i})\mathbb{B}^{(2)}(\bar{u}_{\rm ii}) \cdot f(\bar{u}_{\rm ii}, \bar{u}_{\rm i})$$

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Consider a concrete composite model for which

$$\alpha^{(1)}(z) = 0, \quad \text{if} \quad z \in \bar{v}_{\rm II}$$

$$\alpha^{(2)}(z) = 0, \quad \text{if} \quad z \in \bar{u}_{\rm I}$$

This choice is always possible within the framework of inhomogeneous XXX chain.

## Composite model and scalar products

$$\mathbb{C}(\bar v)\mathbb{B}(\bar u) = \sum \alpha^{(1)}(\bar v_{\mathsf{ii}})\alpha^{(2)}(\bar u_{\mathsf{i}}) f(\bar v_{\mathsf{i}}, \bar v_{\mathsf{ii}}) f(\bar u_{\mathsf{ii}}, \bar u_{\mathsf{i}})$$

$$\times \quad \mathbb{B}^{(1)}(\bar u_{\mathsf{i}})\mathbb{C}^{(1)}(\bar v_{\mathsf{i}}) \quad \mathbb{B}^{(2)}(\bar u_{\mathsf{ii}})\mathbb{C}^{(2)}(\bar v_{\mathsf{ii}})$$

Let $\#\bar v_{\mathsf{i}} = \#\bar u_{\mathsf{i}} = m'$.

## Composite model and scalar products

$$\mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \sum \alpha^{(1)}(\bar{v}_{\mathsf{ii}})\alpha^{(2)}(\bar{u}_{\mathsf{i}})f(\bar{v}_{\mathsf{i}},\bar{v}_{\mathsf{ii}})f(\bar{u}_{\mathsf{ii}},\bar{u}_{\mathsf{i}})$$

$$\times \quad \mathbb{B}^{(1)}(\bar{u}_{\mathsf{i}})\mathbb{C}^{(1)}(\bar{v}_{\mathsf{i}}) \quad \mathbb{B}^{(2)}(\bar{u}_{\mathsf{ii}})\mathbb{C}^{(2)}(\bar{v}_{\mathsf{ii}})$$

Let $\#\bar{v}_{\mathsf{i}} = \#\bar{u}_{\mathsf{i}} = m'$. Then due to the restriction

$$\alpha^{(1)}(z) = 0, \quad \text{if} \quad z \in \bar{v}_{\mathrm{II}}$$

$$\alpha^{(2)}(z) = 0, \quad \text{if} \quad z \in \bar{u}_{\mathrm{I}}$$

we have: $\bar{v}_{\mathsf{ii}} \subset \bar{v}_{\mathrm{I}}$ and $\bar{u}_{\mathsf{i}} \subset \bar{u}_{\mathrm{II}}$ leading to $N - m' \le m$ and $m' \le N - m$. From this we find $m' = N - m$, what implies $\bar{u}_{\mathsf{i}} = \bar{u}_{\mathrm{II}}$, $\bar{v}_{\mathsf{i}} = \bar{v}_{\mathrm{II}}$, $\bar{u}_{\mathsf{ii}} = \bar{u}_{\mathrm{I}}$, and $\bar{v}_{\mathsf{ii}} = \bar{v}_{\mathrm{I}}$.

# Composite model and scalar products

$$\mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \alpha^{(1)}(\bar{v}_{\mathrm{I}})\alpha^{(2)}(\bar{u}_{\mathrm{II}})f(\bar{v}_{\mathrm{II}},\bar{v}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}})$$

$$\times \quad \mathbb{B}^{(1)}(\bar{u}_{\mathrm{II}})\mathbb{C}^{(1)}(\bar{v}_{\mathrm{II}}) \quad \mathbb{B}^{(2)}(\bar{u}_{\mathrm{I}})\mathbb{C}^{(2)}(\bar{v}_{\mathrm{I}})$$

## Composite model and scalar products

$$\mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \alpha^{(1)}(\bar{v}_{\rm I})\alpha^{(2)}(\bar{u}_{\rm II})f(\bar{v}_{\rm II},\bar{v}_{\rm I})f(\bar{u}_{\rm I},\bar{u}_{\rm II})$$

$$\times \quad \mathbb{B}^{(1)}(\bar{u}_{\rm II})\mathbb{C}^{(1)}(\bar{v}_{\rm II}) \quad \mathbb{B}^{(2)}(\bar{u}_{\rm I})\mathbb{C}^{(2)}(\bar{v}_{\rm I})$$

Due to the restriction

$$\alpha^{(1)}(z) = 0, \quad \text{if} \quad z \in \bar{v}_{\rm II}$$

$$\alpha^{(2)}(z) = 0, \quad \text{if} \quad z \in \bar{u}_{\rm I}$$

$$\Rightarrow$$

$$\mathbb{C}^{(1)}(\bar{v}_{\rm II})\mathbb{B}^{(1)}(\bar{u}_{\rm II}) = \alpha^{(1)}(\bar{u}_{\rm II})Z(\bar{v}_{\rm II}|\bar{u}_{\rm II})$$

$$\mathbb{C}^{(2)}(\bar{v}_{\rm I})\mathbb{B}^{(2)}(\bar{u}_{\rm I}) = \alpha^{(2)}(\bar{v}_{\rm I})Z(\bar{u}_{\rm I}|\bar{v}_{\rm I})$$

## Composite model and scalar products

$$\mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \alpha^{(1)}(\bar{v}_{\mathrm{I}})\alpha^{(2)}(\bar{u}_{\mathrm{II}})f(\bar{v}_{\mathrm{II}},\bar{v}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}})$$

$$\times \quad \mathbb{B}^{(1)}(\bar{u}_{\mathrm{II}})\mathbb{C}^{(1)}(\bar{v}_{\mathrm{II}}) \quad \mathbb{B}^{(2)}(\bar{u}_{\mathrm{I}})\mathbb{C}^{(2)}(\bar{v}_{\mathrm{I}})$$

Due to the restriction

$$\alpha^{(1)}(z) = 0, \quad \text{if} \quad z \in \bar{v}_{\mathrm{II}} \qquad \qquad \mathbb{C}^{(1)}(\bar{v}_{\mathrm{II}})\mathbb{B}^{(1)}(\bar{u}_{\mathrm{II}}) = \alpha^{(1)}(\bar{u}_{\mathrm{II}})Z(\bar{v}_{\mathrm{II}}|\bar{u}_{\mathrm{II}})$$

$$\Rightarrow$$

$$\alpha^{(2)}(z) = 0, \quad \text{if} \quad z \in \bar{u}_{\mathrm{I}} \qquad \qquad \mathbb{C}^{(2)}(\bar{v}_{\mathrm{I}})\mathbb{B}^{(2)}(\bar{u}_{\mathrm{I}}) = \alpha^{(2)}(\bar{v}_{\mathrm{I}})Z(\bar{u}_{\mathrm{I}}|\bar{v}_{\mathrm{I}})$$

$$\mathbb{C}(\bar{v})\mathbb{B}(\bar{u}) = \overbrace{\alpha(\bar{v}_{\mathrm{I}})\alpha(\bar{u}_{\mathrm{II}})}^{T}\overbrace{f(\bar{v}_{\mathrm{II}},\bar{v}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}})Z(\bar{u}_{\mathrm{I}}|\bar{v}_{\mathrm{I}})Z(\bar{v}_{\mathrm{II}}|\bar{u}_{\mathrm{II}})}^{R}$$

$$W_{\text{part}}\begin{pmatrix}\bar{v}_{\mathrm{I}} & \bar{u}_{\mathrm{I}}\\ \bar{v}_{\mathrm{II}} & \bar{u}_{\mathrm{II}}\end{pmatrix} = f(\bar{v}_{\mathrm{II}},\bar{v}_{\mathrm{I}})f(\bar{u}_{\mathrm{I}},\bar{u}_{\mathrm{II}})Z(\bar{u}_{\mathrm{I}}|\bar{v}_{\mathrm{I}})Z(\bar{v}_{\mathrm{II}}|\bar{u}_{\mathrm{II}})$$

## Composite model and coproduct property of Bethe vectors

**Exactly** the same method works for the general $\mathfrak{gl}(n)$ case provided we know a formula for the total Bethe vector in the composite model.

$$T(u) = T^{(2)}(u)T^{(1)}(u) \qquad \longrightarrow \qquad T_{i,j}(u) = T^{(2)}_{i,k}(u)T^{(1)}_{k,j}(u)$$

Coproduct

$$\Delta T_{i,j}(u) = T_{k,j}(u) \otimes T_{i,k}(u)$$

$\mathfrak{gl}(n)$ case: $\mathbb{B}(\bar{t}) = \mathcal{P}(\bar{t})|0\rangle$

$$\mathcal{P}(\bar{t}) = \sum_{\text{part}} \frac{\prod_{\mu=1}^{n-1} \alpha_{\mu}^{(2)}(\bar{t}^{\mu}_{\text{I}}) f(\bar{t}^{\mu}_{\text{II}}, \bar{t}^{\mu}_{\text{I}})}{\prod_{\mu=1}^{n-2} f(\bar{t}^{\mu+1}_{\text{II}}, \bar{t}^{\mu}_{\text{I}})} \mathcal{P}(\bar{t}_{\text{I}}) \otimes \mathcal{P}(\bar{t}_{\text{II}})$$

**V. Tarasov, A. Varchenko '95**

**B. Enriquez, S. Khoroshkin, S. Pakuliak '07**

## Composite model and coproduct property of Bethe vectors

$\mathfrak{gl}(n)$

$$\mathbb{C}(\bar{s})\mathbb{B}(\bar{t}) = \sum \left(\prod_{k=1}^{n-1} \alpha_k(\bar{s}^k_{\mathrm{I}})\alpha_k(\bar{t}^k_{\mathrm{II}})\right) W_{\textsf{part}}\begin{pmatrix}\bar{s}_{\mathrm{I}} & \bar{t}_{\mathrm{I}}\\ \bar{s}_{\mathrm{II}} & \bar{t}_{\mathrm{II}}\end{pmatrix}$$

The sum is taken over partitions $\bar{t}^k \Rightarrow \bar{t}^k_{\mathrm{I}} \cup \bar{t}^k_{\mathrm{II}}$ and $\bar{s}^k \Rightarrow \bar{s}^k_{\mathrm{I}} \cup \bar{s}^k_{\mathrm{II}}$ such that $\#\bar{t}^k_{\mathrm{I}} = \#\bar{s}^k_{\mathrm{I}}$, $k = 1, \dots, n-1$. The coefficients $W_{\textsf{part}}$ have the following explicit expressions in terms of the highest coefficients:

$$W_{\textsf{part}}\begin{pmatrix}\bar{s}_{\mathrm{I}} & \bar{t}_{\mathrm{I}}\\ \bar{s}_{\mathrm{II}} & \bar{t}_{\mathrm{II}}\end{pmatrix} = Z(\bar{s}_{\mathrm{I}}|\bar{t}_{\mathrm{I}})Z(\bar{t}_{\mathrm{II}}|\bar{s}_{\mathrm{II}})\, \frac{\prod_{\mu=1}^{n-1} f(\bar{s}^{\mu}_{\mathrm{II}},\bar{s}^{\mu}_{\mathrm{I}})\, f(\bar{t}^{\mu}_{\mathrm{I}},\bar{t}^{\mu}_{\mathrm{II}})}{\prod_{\mu=1}^{n-2} f(\bar{s}^{\mu+1}_{\mathrm{II}},\bar{s}^{\mu}_{\mathrm{I}})\, f(\bar{t}^{\mu+1}_{\mathrm{I}},\bar{t}^{\mu}_{\mathrm{II}})}$$

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## Summary

The coproduct formula for the Bethe vectors immediately leads to the scalar product formula without any additional calculations.
The obtained explicit expression depends on the highest coefficient.

**How to compute the highest coefficients?**

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The coproduct formula for the Bethe vectors immediately leads to the scalar product formula without any additional calculations. The obtained explicit expression depends on the highest coefficient.

**How to compute the highest coefficients?**

There exist relatively simple ways to derive recursions for the highest coefficient. There are at least two types of recursions:

- Recursion with respect to the number of the Bethe parameters
- Recursion with respect to the rank

## Summary

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**How to compute the highest coefficients?**

There exist relatively simple ways to derive recursions for the highest coefficient. There are at least two types of recursions:

- Recursion with respect to the number of the Bethe parameters
- Recursion with respect to the rank

The latter recursion allows one to obtain an explicit representation the highest coefficient. However, this representation is quite involved.

## Summary

Representation for the scalar product in terms of a sum over partitions of the Bethe parameters is not convenient for the applications. However, in the case considered, we deal with the most general case of the scalar product. We can expect certain simplifications in some particular cases.

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Representation for the scalar product in terms of a sum over partitions of the Bethe parameters is not convenient for the applications. However, in the case considered, we deal with the most general case of the scalar product. We can expect certain simplifications in some particular cases.

- The first important particular case is the scalar product involving on-shell Bethe vectors. Then the sums other partitions can be reduced to single determinants for the models described by the $Y(\mathfrak{gl}(3))$, $Y(\mathfrak{gl}(2|1))$, and $U_q(\widehat{\mathfrak{gl}}(3))$ algebras.

## Summary

Representation for the scalar product in terms of a sum over partitions of the Bethe parameters is not convenient for the applications. However, in the case considered, we deal with the most general case of the scalar product. We can expect certain simplifications in some particular cases.

• The first important particular case is the scalar product involving on-shell Bethe vectors. Then the sums other partitions can be reduced to single determinants for the models described by the $Y(\mathfrak{gl}(3))$, $Y(\mathfrak{gl}(2|1))$, and $U_q(\widehat{\mathfrak{gl}}(3))$ algebras.

• The second particular case concerns the models with specific functions $\alpha_i(z)$. For instance, in the $SU(n)$-invariant XXX chain, one has $\alpha_i(z) = 1$ for $i > 1$. This case is almost non-studied.


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