Inhomogeneous quantum quenches in the XXZ spin chain

Jean-Marie Stéphan

1 Camille Jordan Institute, University of Lyon 1, Villeurbanne, France

Correlation functions of quantum integrable systems and beyond, 60th birthday of Jean-Michel Maillet

JMS [arXiv:1707.06625]
see also
J. Dubail, JMS, and P. Calabrese [Scipost Physics 2017]
J. Dubail, JMS, J. Viti, and P. Calabrese [Scipost Physics 2017]
Outline

1. Inhomogeneous Quantum Quenches
2. An exact formula for the return probability
3. Discussion
Quantum quenches

\[ H(\lambda) \]

Prepare a system in some pure state \(|\Psi_0\rangle\)

Evolve with \(H(\lambda)\)

\[ |\Psi(t)\rangle = e^{-iH(\lambda)t} |\Psi_0\rangle \]

Unitary evolution, no coupling to an environment.
Integrable systems

- 2d statistical mechanics.

Integrable models are good representatives of universality classes (e.g. Ising model, six-vertex model, etc).

- 1d out of equilibrium quantum dynamics

Peculiar thermalization properties.
May be realized experimentally in cold atom systems, [Kinoshita, Wenger & Weiss, Nat. 2006]
Quench studied here

Initial state \[ |\Psi_0\rangle = |\ldots \uparrow \uparrow \uparrow \uparrow \uparrow \downarrow \ldots \rangle \]

Time evolution \[ |\Psi(t)\rangle = e^{-itH_{XXZ}} |\Psi_0\rangle \]

\[ H_{XXZ} = \sum_{x \in \mathbb{Z}+1/2} \left( S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3 \right) \]

Free fermion case (\(\Delta = 0\)) [Antal, Racz, Rakos, and Schütz, 1999]

Interactions: MPS techniques (numerics)

[Gobert, Kollath, Schollwöck, and Schütz 2005]

Works nicely because growth of entanglement is \(S(t) \approx \log t\).
Finite speed of propagation: light cone.

Regime: large $x$, large $t$, finite $x/t$.

Density profile:

Widely available libraries today [http://itensor.org]
Inhomogeneous Quantum Quenches

An exact formula for the return probability

Discussion

Effective descriptions

- Generalized hydrodynamics (ballistic)
  
  [Castro-Alvaredo, Doyon, Yoshimura 2016]
  
  [Bertini, Collura, De Nardis, Fagotti 2016]

  This particular quench (e. g. $\Delta = 1/2$),

  $$S^3_x(x/t) = -\frac{2}{\pi} \arcsin \frac{x}{t}$$

  [De Luca, Collura, Viti 2017]

- What about $\Delta = 1$, where super diffusive behavior was conjectured? [Ljubotina, Znidaric, Prosen 2017]
Inhomogeneous quantum systems

[Dubail, JMS, Calabrese 2017]...

\[ H = \sum_{j=1}^{L} f(j/L)h_j, \quad h_j \text{ local Hamiltonian density.} \]

Might want do write some simple field theory action

\[ S = \frac{1}{4\pi K} \int dzd\bar{z}e^{\sigma(z,\bar{z})}(\partial_z \varphi)(\partial_{\bar{z}} \varphi) \]

Relevant to quantum gases in traps, etc.
I will compute the return probability $\mathcal{R}(t) = |\langle \Psi(0)|\Psi(t)\rangle|^2$

Simple guess for asymptotics: ballistic, so $\mathcal{R}(t) \sim e^{-at}$.

Nb: $\overline{\mathcal{R}(t)} \sim \prod_{k=1}^{\infty} (1 - e^{-2k\eta})^2$, $\cosh \eta = \Delta > 1$

[Mossel, Caux 2011]
Fun with dimers
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Fun with dimers
Correlations functions: gaussian free field, or coulomb gas, or free compact boson CFT \((c = 1)\), or euclidean Luttinger liquid.
Dimer coverings on the Aztec diamond
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Arctic circle theorem  [Jockusch, Propp and Shor 1998]
Fits into curved CFT formalism [Allegra, Dubail, JMS, Viti 2016]
Can add interaction between dimers (no theorem)
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Six-vertex model

\[ a = d \sin(\gamma + \epsilon) \quad , \quad b = d \sin \epsilon \quad , \quad c = d \sin \gamma \]

\[ \Delta = \frac{a^2 + b^2 - c^2}{2ab} = \cos \gamma. \]
An Observation

\[ b = 1 \]

\[ \frac{\tau}{2} - \frac{\tau}{2} \]

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]
An Observation

\[ b = \frac{1}{2} \]
An Observation

[JMS 2014]  [Allegra, Dubail, JMS, Viti 2016]

\[ b \to 0 \]

\[ \frac{-\tau}{2} \]

\[ \frac{-\tau}{2} \]
An Observation

[JMS 2017]
Familiar from e. g Quantum transfer matrix approach. [Wuppertal].
Similar calculation for the Néel state [Piroli, Pozsgay, Vernier 2017]

\[ Z(\tau) = \lim_{n \to \infty} Z(a = 1, b = \frac{\tau}{2n}, \Delta) \]

Considered by [Korepin 1982]. Determinant formula [Izergin 1987]

\[ Z = \frac{[\sin \epsilon]^{n^2}}{\prod_{k=0}^{n-1} k!^2} \det_{0 \leq i, j \leq n-1} \left( \int_{-\infty}^{\infty} du \, u^{i+j} e^{-\epsilon u} \frac{1 - e^{-\gamma u}}{1 - e^{-\pi u}} \right) \]

Put this in a more tractable form [Slavnov, J. Math. Sci 2003]
(see also [Colomo Pronko 2003])
Hankel matrices and orthogonal polynomials

- Choose a scalar product \( \langle f, g \rangle = \int dx f(x)g(x)w(x) \)
- Let \( \{p_k(x)\}_{k \geq 0} \) be a set of monic orthogonal for the scalar product , \( \langle p_k, p_l \rangle = h_k \delta_{kl} \)
- Consider the Hankel matrix \( A \), with elements \( A_{ij} = \langle x^{i+j} \rangle \)

\[
\det A = \prod_{k=0}^{n-1} h_k, \quad (A^{-1})_{ij} = \left. \frac{\partial^{i+j} K_n(x, y)}{i! j! \partial x^i \partial y^j} \right|_{x=0, y=0} \quad \text{with}
\]

\[
K_n(x, y) = \sum_{k=0}^{n-1} \frac{p_k(x)p_k(y)}{h_k} = \frac{1}{h_{n-1}} \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y}
\]
Laguerre polynomials

\[ w(x) = e^{-\epsilon x} \text{ on } \mathbb{R}_+, \quad \det(A) = \frac{\prod_{k=0}^{n-1} k!^2}{\epsilon n^2} \]

\[ Z = \left( \frac{\sin \epsilon}{\epsilon} \right)^{n^2} \times \frac{\det}{0 \leq i, j \leq n-1} \left( \int_{-\infty}^{\infty} du \frac{u^i e^{-\epsilon u}}{1 - e^{-\gamma u}} \frac{1 - e^{-\gamma u}}{1 - e^{-\pi u}} \right) \]

Now use \( \frac{\det A}{\det B} = \det(B^{-1}A) = \det(1 + B^{-1}(A - B)) \) to get something well behaved.
Fredholm determinant

\[ Z(\tau) = \langle e^{\tau H} \rangle = e^{-\frac{1}{24} (\tau \sin \gamma)^2} \det(I - V) \]

\[ V(x, y) = B_0(x, y) \omega(y) \]

\[ B_\alpha(x, y) = \frac{\sqrt{y} J_\alpha(\sqrt{x}) J'_\alpha(\sqrt{y}) - \sqrt{x} J_\alpha(\sqrt{y}) J'_\alpha(\sqrt{x})}{2(x - y)} \]

\[ \omega(y) = \Theta(y) - \frac{1 - e^{-\gamma y/(2\tau \sin \gamma)}}{1 - e^{-\pi y/(2\tau \sin \gamma)}} \]

\[ \log \det(I - V) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int dx_1 \ldots dx_k V(x_1, x_2) \ldots V(x_k, x_1) \]
Area law and arctic curves

\[
x(s) = \frac{\sin s \sin(\gamma + s)}{\tau} \left[ \alpha^2 \csc^2 \alpha s \left\{ \cos(2\gamma + 3s)(\cos s - \alpha \sin s \cot \alpha s) + \alpha \sin \right. \right. \\
\left. \left. \sin^2(\gamma + s) + \sin^2 s \right] \right.
\]

\[
y(s) = \frac{\sin^2(\gamma + s)}{\tau} \left[ 2\alpha^2 \csc \gamma \sin^2 s \csc^2 \alpha s \left\{ 2\alpha \sin s \cot \alpha s \sin(\gamma + s) - \sin(\gamma + \\
\sin^2(\gamma + s) + \sin^2 s \right] \right.
\]

[Colomo, Pronko 2009]
Asymptotics

Easiest: use [Zinn-Justin 2000] [Bleher, Fokin 2006]

\[ Z(\tau) \underset{\tau \to \infty}{\sim} \exp \left( \left[ \frac{\pi^2}{(\pi - \gamma)^2} - 1 \right] \frac{(\tau \sin \gamma)^2}{24} \right) \tau^{\kappa(\gamma)} O(1) \]

\[ \kappa(\gamma) = \frac{1}{12} - \frac{(\pi - \gamma)^2}{6\pi \gamma} \]

Interpretation: free energy of the fluctuating region.
Back to real time

Analytic continuation

- Return probability: \( \tau = it \)
- Correlations: \( y = it \) and \( \tau \to 0^+ \)

Continuation of the arctic curves should give the light cone:

Free fermions: \( x^2 + y^2 = (\tau/2)^2 \quad \longrightarrow \quad x = \pm t \)
Interactions: complicated \( \longrightarrow \quad x = \pm (\sin \gamma)t = \pm \sqrt{1 - \Delta^2 t} \)

This coincides exactly with the result of generalized hydrodynamics
Analytic continuation

Numerical observations (huge precision, $t$ up to 600):

- Root of unity, $\gamma = \frac{\pi p}{q}$

\[-\log R(t) = \left(\frac{q^2}{(q-1)^2} - 1\right) \frac{(t \sin \gamma)^2}{12} + O(\log t)\]

Coincides with analytic continuation only when $p = 1$.

- Non root of unity

\[-\log R(t) = t \sin \gamma + O(\log t)\]
Analytic continuation

Numerical observations (huge precision, $t$ up to 600):

![Graph showing numerical observations]

Compatible also with [De Luca, Collura, Viti 2017]

How about a proof using Riemann-Hilbert techniques?

[Its, Izergin, Korepin, Slavnov 1990]
The special case $\Delta = 1$

$$\mathcal{R}(t) = \left| \det(I - K) \right|^2 \text{ on } L^2([0; \sqrt{t}]).$$

$$K(u, v) = i\sqrt{u}\sqrt{v}e^{-\frac{1}{2}i(u^2+v^2)}J_0(uv) \rightarrow \frac{e^{i\pi/4}}{\sqrt{2\pi}}e^{-\frac{i}{2}(u-v)^2}$$

Then, computing $\text{Tr}K^n$ asymptotically is much easier.

Final Result:

$$\mathcal{R}(t) \sim \exp \left( -\zeta(3/2)\sqrt{t/\pi} \right) t^{1/2}O(1)$$

By the previous logic, transport should be diffusive for this quench.
Remark on subleading corrections

Careful when extracting the exponent!

Similar analysis in [Misguich, Mallick, Krapivsky 2017], numerically supporting diffusive behavior
Application: Entanglement entropy

\[ \rho_A = \text{Tr}_B |\Psi(t)\rangle \langle \Psi(t)|, \quad S = -\text{Tr} \rho_A \log \rho_A. \]

**\( \Delta = 0 \):** Easy in CFT, provided the density profile is known
[Dubail, JMS, Viti, Calabrese 2017]

\[ S(x, t) = \frac{1}{6} \log \left( t \left[ 1 - \frac{x^2}{t^2} \right]^{3/2} \right) + \text{cst}, \quad t > x \]

Guessed earlier from numerics [Eisler and Peschel 2014]

**\( \Delta \neq 0 \):**

\[ S(x, t) = \frac{1}{6} \log(tf(x/t)) + \text{cst} \]
What about non integrable? (but still $U(1)$)

Is there a relation with toy models of random quantum circuits?

[Nahum, Vijay, Haah 2017], [Nahum, Ruhman, Huse 2017]
[von Keyserlingk, Rakovsky, Pollmann, Sondhi 2017]
Conclusion

- Exact determinant formula for the return probability.

- Other computations with Quantum inverse scattering?

- Intricacies of the analytic continuation $\tau \rightarrow it$.

- Transport at $\Delta = 1$ should be diffusive.

- Integrable vs non Integrable
Happy birthday Jean-Michel!