# Inhomogeneous quantum quenches in the XXZ spin chain

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Correlation functions of quantum integrable systems and beyond, 60th birthday of Jean-Michel Maillet

JMS [arXiv:1707.06625]
see also
J. Dubail, JMS, and P. Calabrese [Scipost Physics 2017]
J. Dubail, JMS, J. Viti, and P. Calabrese [Scipost Physics 2017]
N. Allegra, J. Dubail, JMS and J. Viti [J. Stat. Mech 2016]

# Outline

# Quantum quenches

$H(\lambda)$

Prepare a system in some pure state $|\Psi_0\rangle$

Evolve with $H(\lambda)$

$$|\Psi(t)\rangle = e^{-iH(\lambda)t}\,|\Psi_0\rangle$$

Unitary evolution, no coupling to an environment.

# Integrable systems

- 2d statistical mechanics.

  Integrable models are good representatives of universality classes (e. g. Ising model, six-vertex model, etc).

- 1d out of equilibrium quantum dynamics

  Peculiar thermalization properties.
  May be realized experimentally in cold atom systems, [Kinoshita, Wenger & Weiss, Nat. 2006]

# Quench studied here

Initial state $|\Psi_0\rangle = |\ldots\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\ldots\rangle$

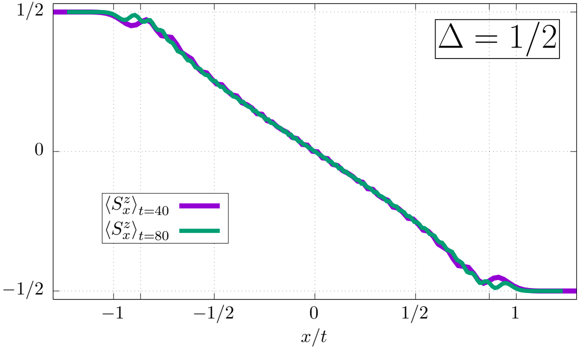
Time evolution $|\Psi(t)\rangle = e^{-itH_{XXZ}}|\Psi_0\rangle$

$$H_{XXZ} = \sum_{x\in\mathbb{Z}+1/2}\left(S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta S_x^3 S_{x+1}^3\right)$$

Free fermion case $(\Delta = 0)$ [Antal, Racz, Rakos, and Schütz, 1999]

Interactions: MPS techniques (numerics)
[Gobert, Kollath, Schollwöck, and Schütz 2005]
Works nicely because growth of entanglement is $S(t) \approx \log t$.

- Finite speed of propagation: light cone.
- Regime: large $x$, large $t$, finite $x/t$.
- Density profile:




Widely available libraries today [http://itensor.org]

# Effective descriptions

- Generalized hydrodynamics (ballistic)
  [Castro-Alvaredo, Doyon, Yoshimura 2016]
  [Bertini, Collura, De Nardis, Fagotti 2016]

  This particular quench (e. g. $\Delta = 1/2$),

  $$S_x^3(x/t) = -\frac{2}{\pi}\arcsin\frac{x}{t}$$

  [De Luca, Collura, Viti 2017]

- What about $\Delta = 1$, where super diffusive behavior was conjectured? [Ljubotina, Znidaric, Prosen 2017]

# Inhomogeneous quantum systems

[Dubail, JMS, Calabrese 2017]. . .

$$H = \sum_{j=1}^{L} f(j/L) h_j \quad , \quad h_j \text{ local Hamiltonian density.}$$

Might want do write some simple field theory action

$$\mathcal{S} = \frac{1}{4\pi K} \int dz d\bar{z} e^{\sigma(z,\bar{z})} (\partial_z \varphi)(\partial_{\bar{z}} \varphi)$$

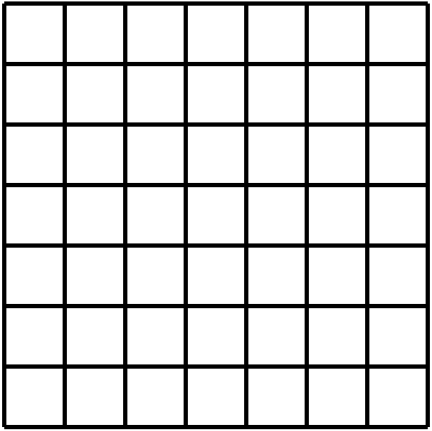
Relevant to quantum gases in traps, etc.

I will compute the return probability $\mathcal{R}(t) = |\langle\Psi(0)|\Psi(t)\rangle|^2$

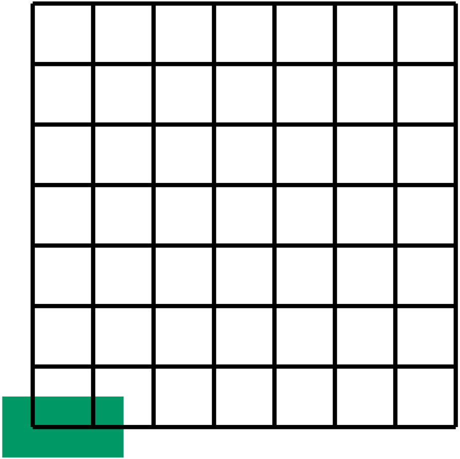
Simple guess for asymptotics: ballistic, so $\mathcal{R}(t) \sim e^{-at}$.

Nb: $\overline{\mathcal{R}(t)} \sim \prod_{k=1}^{\infty} \left(1 - e^{-2k\eta}\right)^2 \quad , \quad \cosh\eta = \Delta > 1$

[Mossel, Caux 2011]



Fun with dimers



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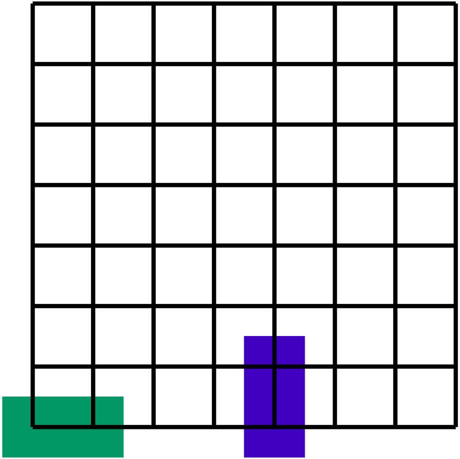

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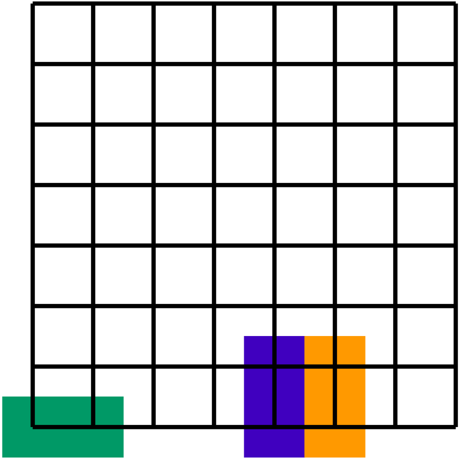

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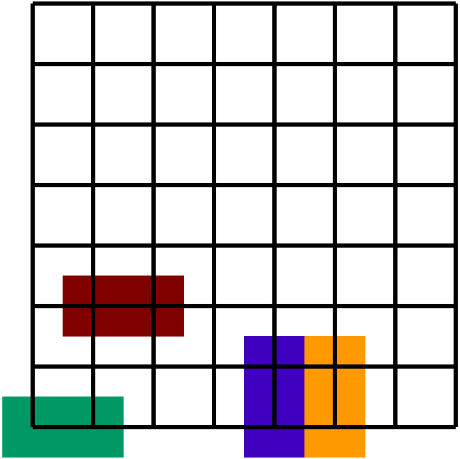

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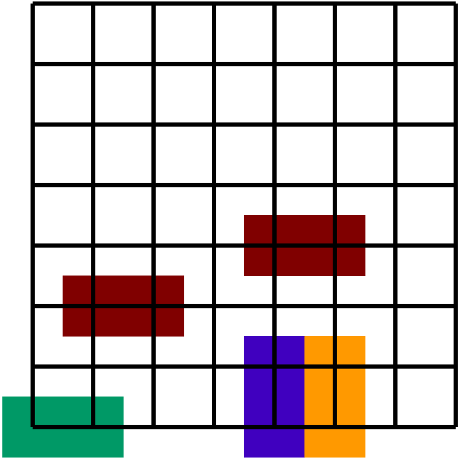

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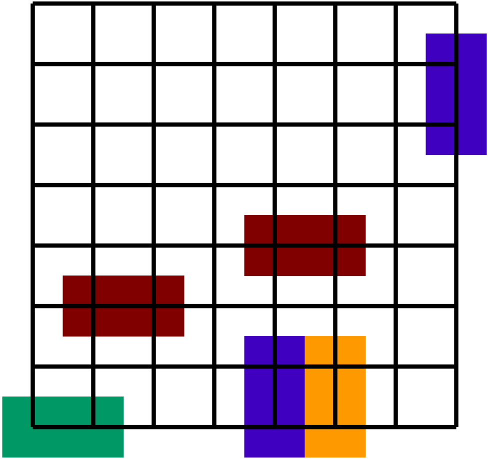

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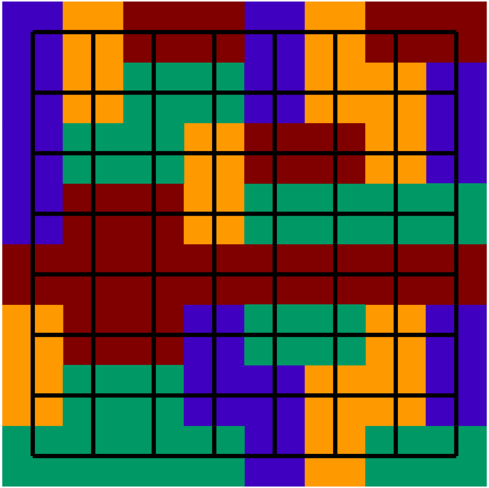

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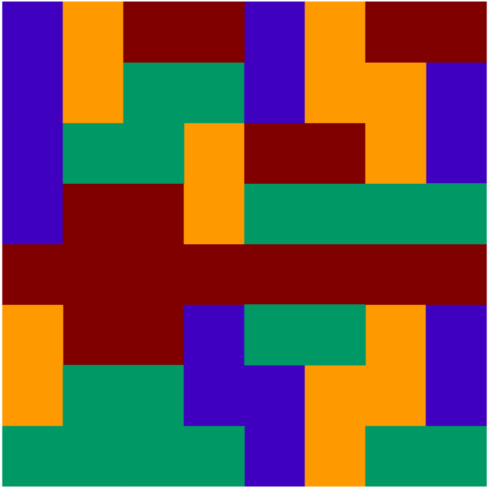

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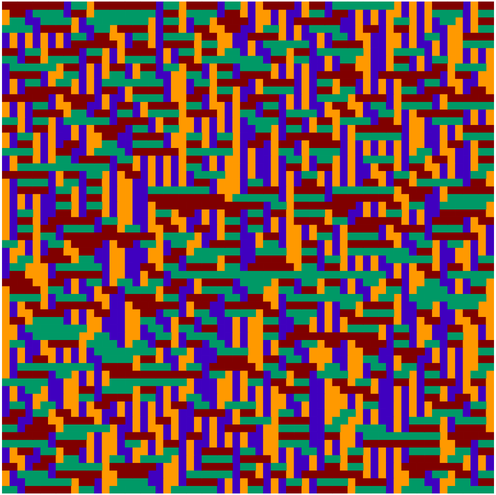

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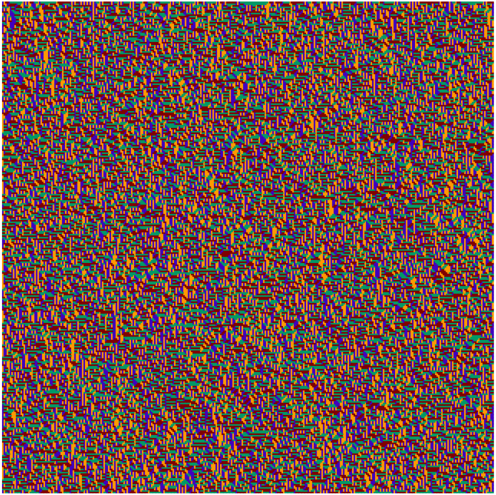
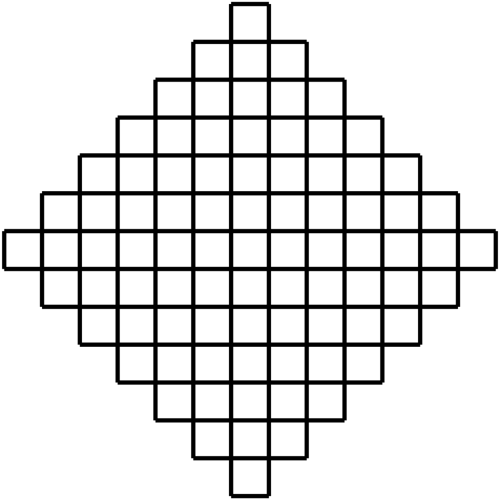


Correlations functions: gaussian free field, or coulomb gas, or free compact boson CFT ($c = 1$), or euclidean Luttinger liquid.

# Dimer coverings on the Aztec diamond



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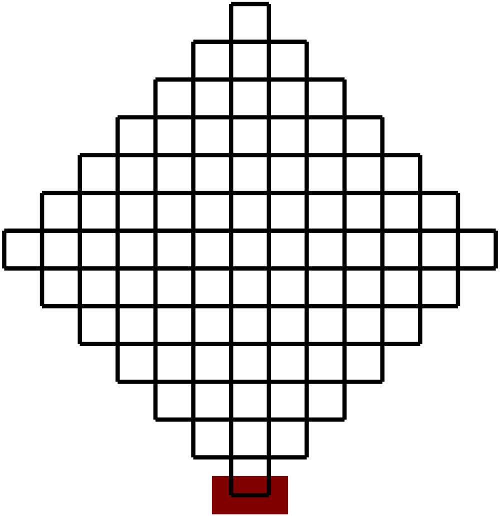

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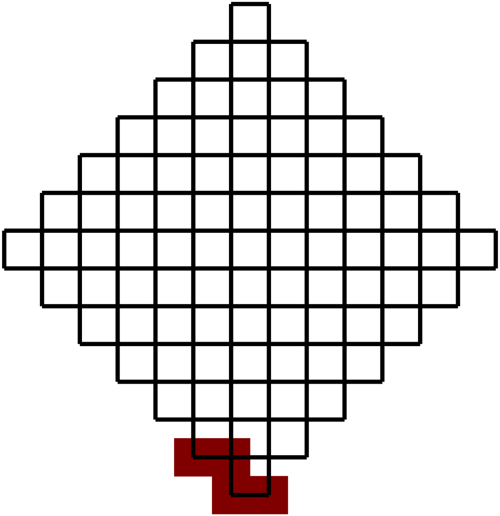

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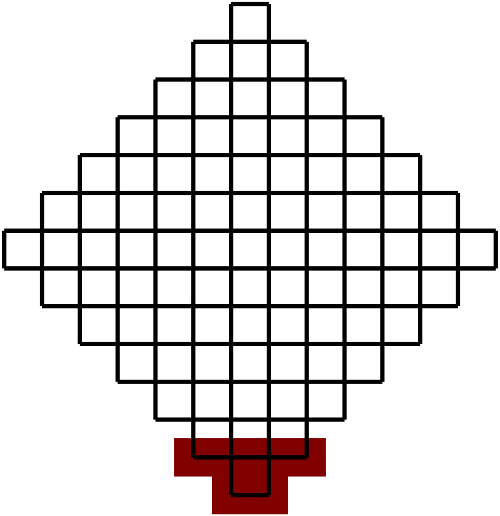

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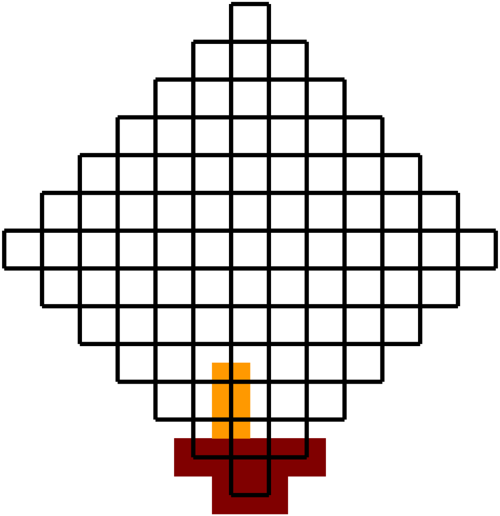

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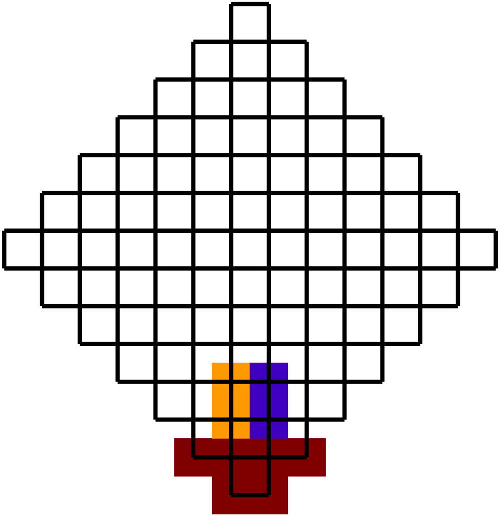

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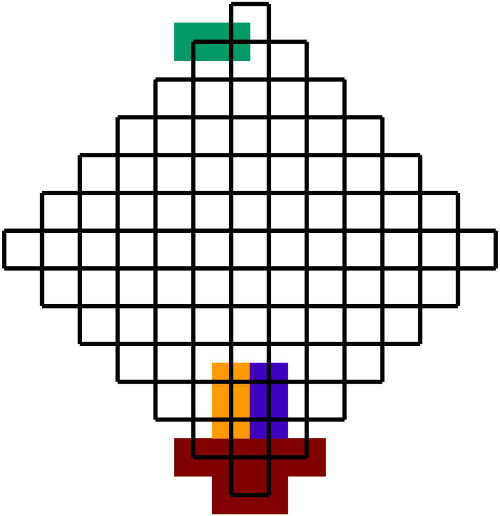

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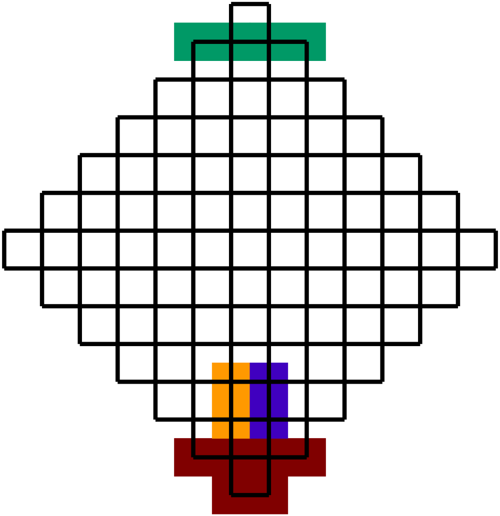

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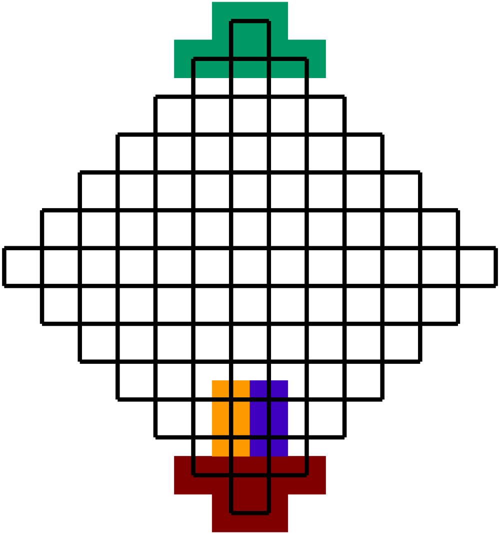

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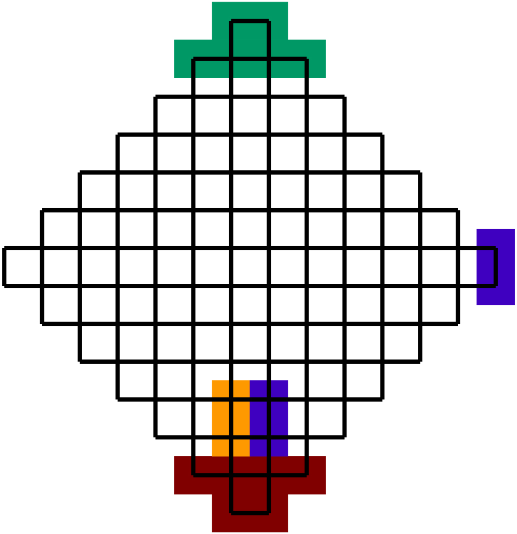

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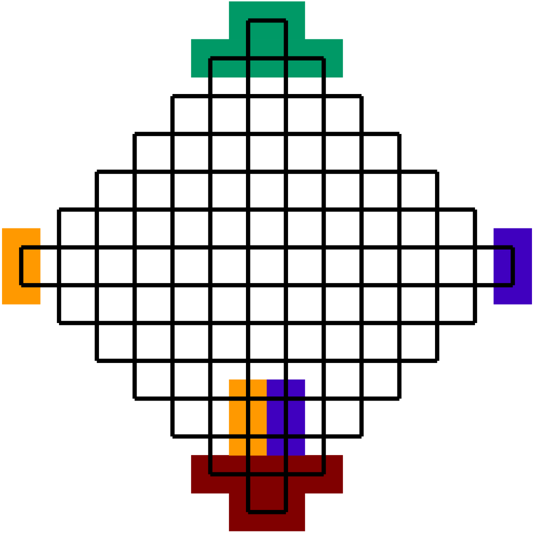

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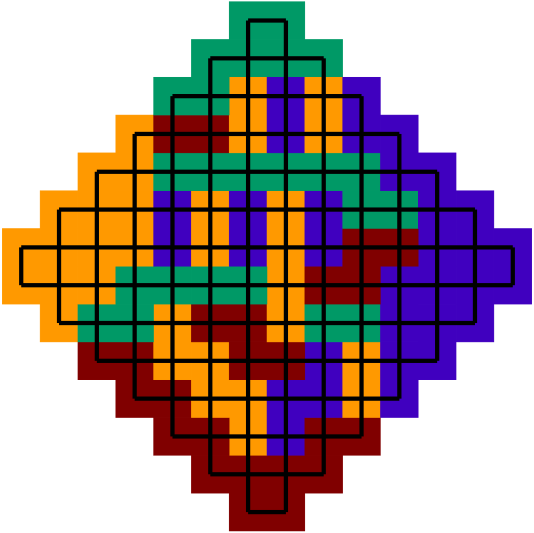

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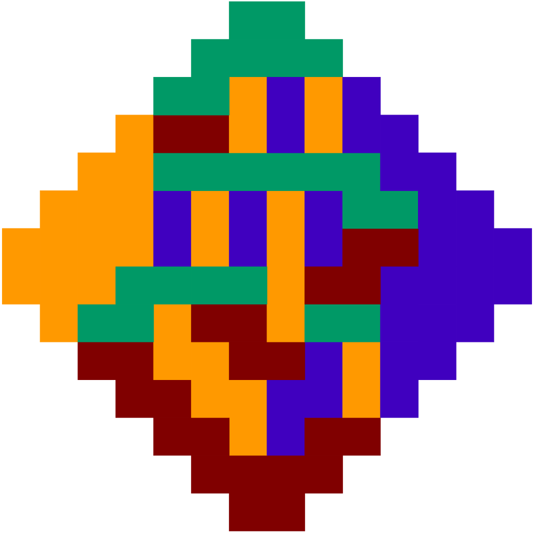
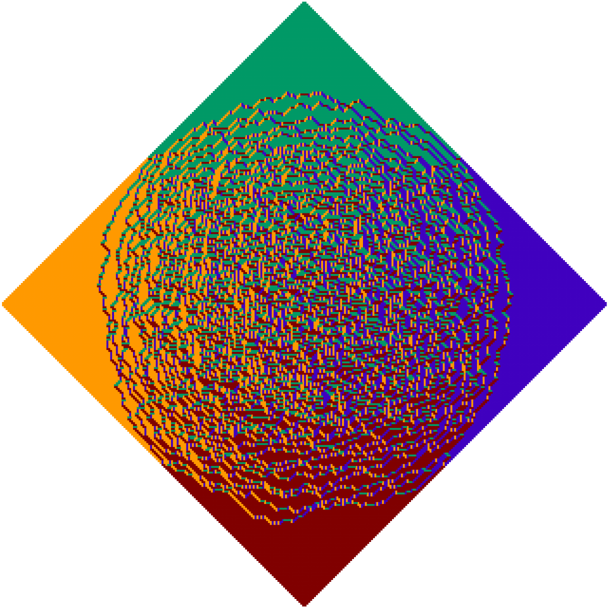
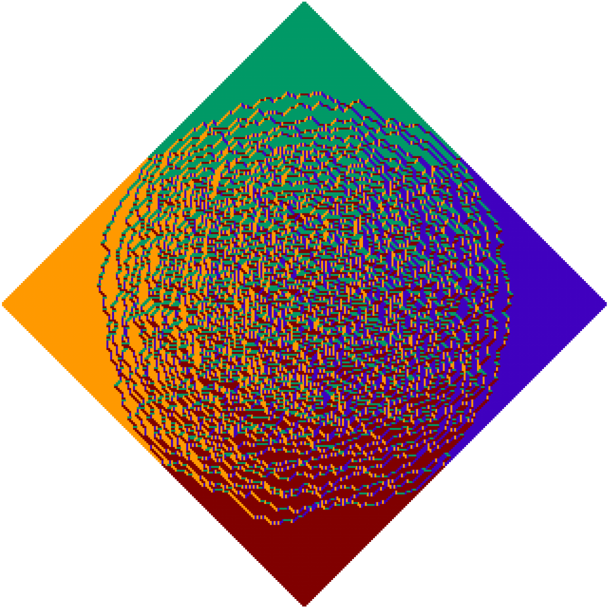
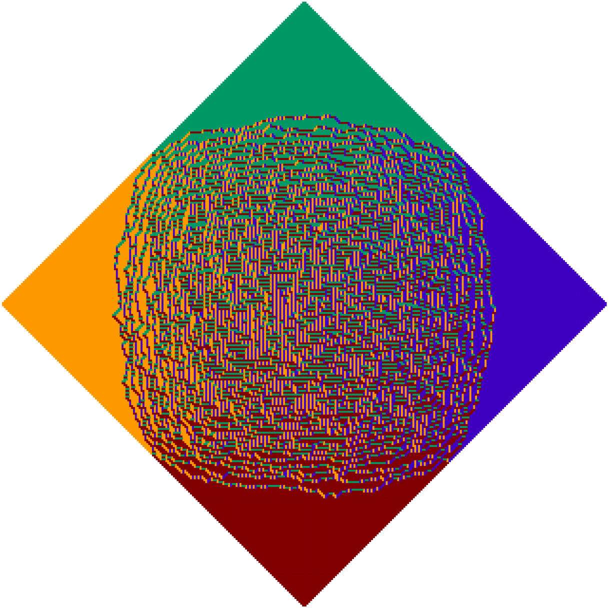




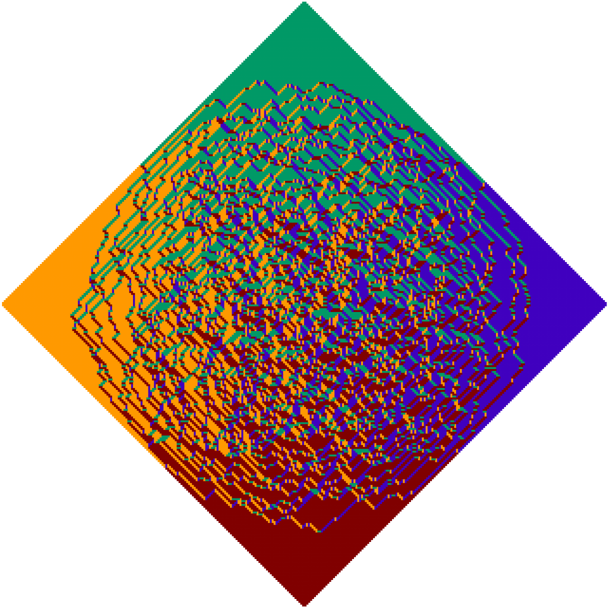
Arctic circle theorem [Jockusch, Propp and Shor 1998]



Fits into curved CFT formalism [Allegra, Dubail, JMS, Viti 2016]

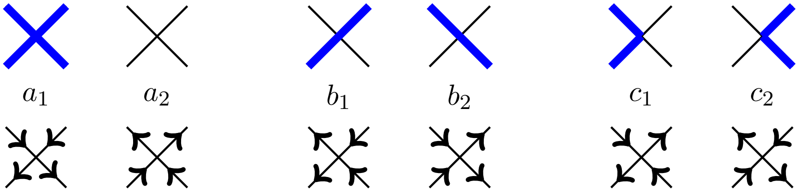


Can add interaction between dimers (no theorem)



Can add interaction between dimers (no theorem)

# Six-vertex model

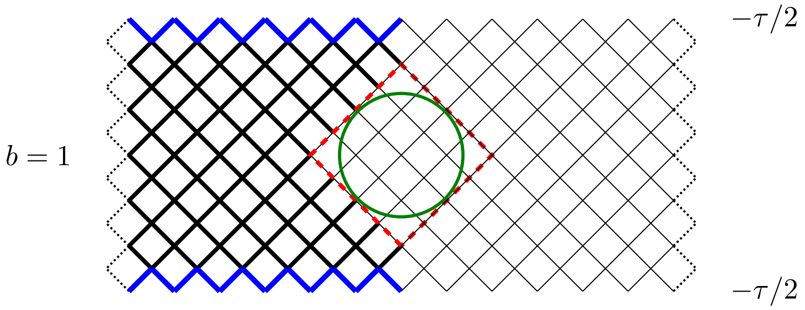



$$a = d\sin(\gamma+\epsilon) \quad , \quad b = d\sin\epsilon \quad , \quad c = d\sin\gamma$$

$$\Delta = \frac{a^2+b^2-c^2}{2ab} = \cos\gamma.$$

# An Observation

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]



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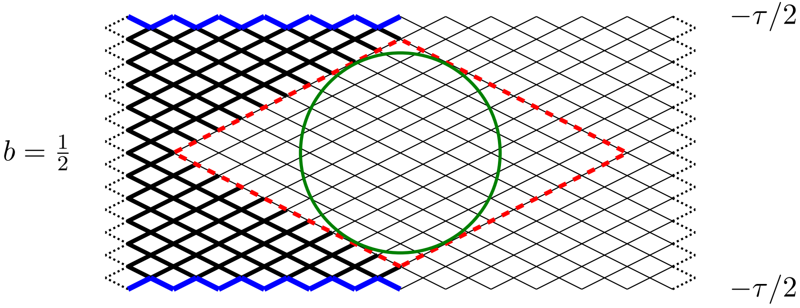
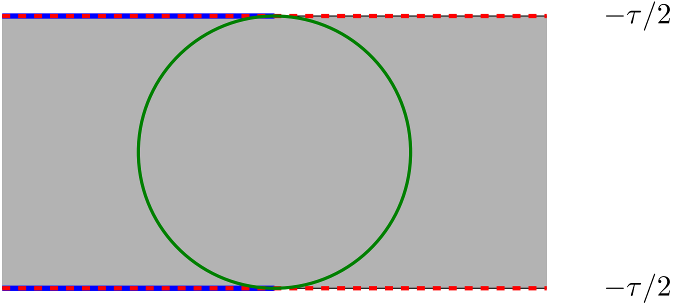


# An Observation

[JMS 2014] [Allegra, Dubail, JMS, Viti 2016]

$b \to 0$



# An Observation

[JMS 2017]

Familiar from e. g Quantum transfer matrix approach. [Wuppertal].
Similar calculation for the Néel state [Piroli, Pozsgay, Vernier 2017]

$$\mathcal{Z}(\tau) = \lim_{n\to\infty} Z(a=1, b=\frac{\tau}{2n}, \Delta)$$

Considered by [Korepin 1982]. Determinant formula [Izergin 1987]

$$Z = \frac{[\sin\epsilon]^{n^2}}{\prod_{k=0}^{n-1} k!^2} \det_{0\le i,j\le n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u} \frac{1-e^{-\gamma u}}{1-e^{-\pi u}}\right)$$

Put this in a more tractable form [Slavnov, J. Math. Sci 2003]
(see also [Colomo Pronko 2003])

# Hankel matrices and orthogonal polynomials

- Choose a scalar product $\langle f,g\rangle = \int dx f(x)g(x)w(x)$
- Let $\{p_k(x)\}_{k\geq 0}$ be a set of monic orthogonal for the scalar product , $\langle p_k,p_l\rangle = h_k\delta_{kl}$
- Consider the Hankel matrix $A$, with elements $A_{ij}=\langle x^{i+j}\rangle$

$$\det A = \prod_{k=0}^{n-1} h_k \quad , \quad (A^{-1})_{ij} = \left.\frac{\partial^{i+j}K_n(x,y)}{i!j!\partial x^i\partial y^j}\right|_{\substack{x=0\\y=0}} \text{ with}$$

$$K_n(x,y) = \sum_{k=0}^{n-1}\frac{p_k(x)p_k(y)}{h_k} = \frac{1}{h_{n-1}}\frac{p_n(x)p_{n-1}(y)-p_{n-1}(x)p_n(y)}{x-y}$$

# Laguerre polynomials

$$w(x) = e^{-\epsilon x} \text{ on } \mathbb{R}_+ \quad , \quad \det(A) = \frac{\prod_{k=0}^{n-1} k!^2}{\epsilon^{n^2}}$$

$$Z = \left(\frac{\sin\epsilon}{\epsilon}\right)^{n^2} \times \frac{\det\limits_{0\le i,j\le n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u}\frac{1-e^{-\gamma u}}{1-e^{-\pi u}}\right)}{\det\limits_{0\le i,j\le n-1}\left(\int_{-\infty}^{\infty} du\, u^{i+j} e^{-\epsilon u}\Theta(u)\right)}$$

Now use $\frac{\det A}{\det B} = \det(B^{-1}A) = \det(1 + B^{-1}(A-B))$ to get something well behaved.

# Fredholm determinant

$$\mathcal{Z}(\tau) = \langle e^{\tau H} \rangle = e^{-\frac{1}{24}(\tau \sin\gamma)^2} \det(I - V)$$

$$V(x,y) = B_0(x,y)\,\omega(y)$$

$$B_\alpha(x,y) = \frac{\sqrt{y} J_\alpha(\sqrt{x}) J'_\alpha(\sqrt{y}) - \sqrt{x} J_\alpha(\sqrt{y}) J'_\alpha(\sqrt{x})}{2(x-y)}$$

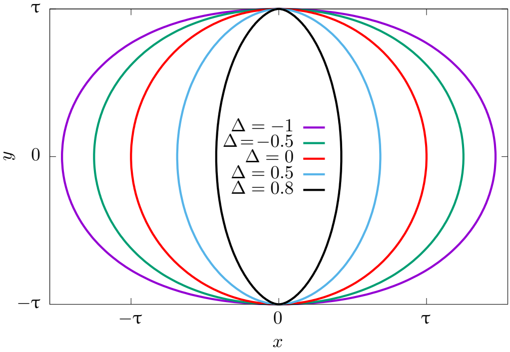
$$\omega(y) = \Theta(y) - \frac{1 - e^{-\gamma y/(2\tau \sin\gamma)}}{1 - e^{-\pi y/(2\tau \sin\gamma)}}$$

$$\log\det(I - V) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \int dx_1 \dots dx_k V(x_1, x_2) \dots V(x_k, x_1)$$

# Area law and arctic curves

$$\frac{x(s)}{\tau} = \frac{\sin s \sin(\gamma + s) \left[\alpha^2 \csc^2 \alpha s \left\{\cos(2\gamma + 3s)(\cos s - \alpha \sin s \cot \alpha s) + \alpha \sin\right.\right.}{\sin^2(\gamma + s) + \sin^2 s}$$

$$\frac{y(s)}{\tau} = \frac{\sin^2(\gamma + s) \left[2\alpha^2 \csc \gamma \sin^2 s \csc^2 \alpha s \left\{2\alpha \sin s \cot \alpha s \sin(\gamma + s) - \sin(\gamma +\right.\right.}{\sin^2(\gamma + s) + \sin^2 s}$$




[Colomo, Pronko 2009]

# Asymptotics

Easiest: use [Zinn-Justin 2000] [Bleher, Fokin 2006]

$$\mathcal{Z}(\tau) \underset{\tau\to\infty}{\sim} \exp\left(\left[\frac{\pi^2}{(\pi-\gamma)^2}-1\right]\frac{(\tau\sin\gamma)^2}{24}\right)\tau^{\kappa(\gamma)}O(1)$$

$$\kappa(\gamma)=\frac{1}{12}-\frac{(\pi-\gamma)^2}{6\pi\gamma}$$

Interpretation: free energy of the fluctuating region.

# Back to real time

**Analytic continuation**

- Return probability: $\tau = it$
- Correlations: $y = it$ and $\tau \to 0^+$

Continuation of the arctic curves should give the light cone:

Free fermions: $x^2 + y^2 = (\tau/2)^2 \quad \longrightarrow \quad x = \pm t$
Interactions: $\text{complicated} \quad \longrightarrow \quad x = \pm(\sin\gamma)t = \pm\sqrt{1-\Delta^2}\, t$

This coincides exactly with the result of generalized hydrodynamics

# Analytic continuation

Numerical observations (huge precision, $t$ up to $600$):

- Root of unity, $\gamma = \frac{\pi p}{q}$

$$-\log \mathcal{R}(t) = \left( \frac{q^2}{(q-1)^2} - 1 \right) \frac{(t \sin \gamma)^2}{12} + O(\log t)$$

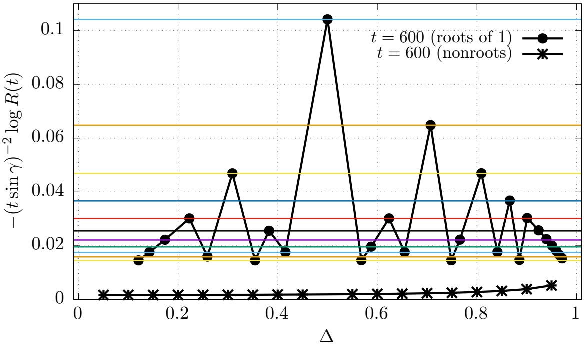
  Coincides with analytic continuation only when $p = 1$.

- non root of unity

$$-\log \mathcal{R}(t) = t \sin \gamma + O(\log t)$$

# Analytic continuation

Numerical observations (huge precision, $t$ up to $600$):




Compatible also with [De Luca, Collura, Viti 2017]

How about a proof using Riemann-Hilbert techniques?

[Its, Izergin, Korepin, Slavnov 1990]

# The special case $\Delta = 1$

$\mathcal{R}(t) = |\det(I-K)|^2$ on $L^2([0;\sqrt{t}])$.

$$K(u,v) = i\sqrt{u}\sqrt{v}e^{-\frac{1}{2}i\left(u^2+v^2\right)}J_0(uv) \quad \longrightarrow \quad \frac{e^{i\pi/4}}{\sqrt{2\pi}}e^{-\frac{i}{2}(u-v)^2}$$

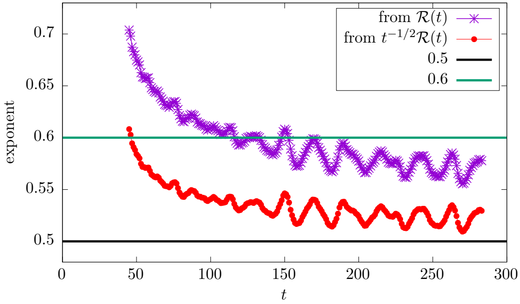
Then, computing $\mathrm{Tr}K^n$ asymptotically is much easier.

Final Result:

$$\mathcal{R}(t) \sim \exp\left(-\zeta(3/2)\sqrt{t/\pi}\right) t^{1/2} O(1)$$

By the previous logic, transport should be diffusive for this quench.

# Remark on subleading corrections

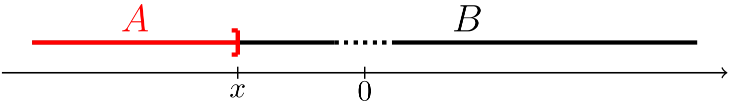



Careful when extracting the exponent!

Similar analysis in [Misguich, Mallick, Krapivsky 2017], numerically supporting diffusive behavior

# Application: Entanglement entropy

$$\rho_A = \mathrm{Tr}_B \, |\Psi(t)\rangle \langle \Psi(t)| \quad , \quad S = -\mathrm{Tr}\, \rho_A \log \rho_A .$$




$\underline{\Delta = 0}$: Easy in CFT, provided the density profile is known

[Dubail, JMS, Viti, Calabrese 2017]

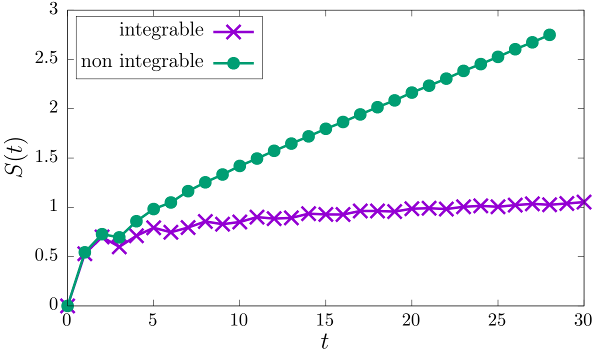
$$S(x,t) = \frac{1}{6} \log \left( t \left[ 1 - x^2/t^2 \right]^{3/2} \right) + \mathrm{cst} \qquad , \qquad t > x$$

Guessed earlier from numerics [Eisler and Peschel 2014]

$\underline{\Delta \neq 0}$:

$$S(x,t) = \frac{1}{6} \log(t f(x/t)) + \mathrm{cst}$$

# What about non integrable? (but still $U(1)$)




Is there a relation with toy models of random quantum circuits?

[Nahum, Vijay, Haah 2017], [Nahum, Ruhman, Huse 2017]

[von Keyserlingk, Rakovsky, Pollmann, Sondhi 2017]

# Conclusion

- Exact determinant formula for the return probability.
- Other computations with Quantum inverse scattering?
- Intricacies of the analytic continuation $\tau \to it$.
- Transport at $\Delta = 1$ should be diffusive.
- Integrable vs non Integrable

Happy birthday Jean-Michel!