Putting generalised Toda QFT on the lattice

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Based on joint work with D. Ridout and C. Meneghelli
(arXiv:1504.04572)

Theory of quantum integrable models $\subset$ quantum group theory

For large classes of quantum integrable models

$(\mathcal{O}, \mathcal{H}, \mathcal{C})$: *Algebra of observables* $\mathcal{O}$, *Hilbert space* $\mathcal{H}$,
$\mathcal{C}$: *Set of conserved quantities containing Hamiltonian* H.

Example: XXZ:

- $\mathcal{O}$: *Generated by local spins* S_n^a, $a = \pm, 0$, $n = 1, \ldots, N$.
- $\mathcal{H} = \otimes_{n=1}^{N} \mathbb{C}^{2j+1}$.
- *From* $\mathrm{Tr}(\mathsf{M}(\lambda)) = \sum_k \lambda^k \mathsf{C}_k$, $\mathsf{M}(\lambda) = \mathsf{L}_N(\lambda) \cdots \mathsf{L}_1(\lambda)$.

there exists a quasi-triangular Hopf-algebra $(\mathcal{A}, \mathcal{R})$

$\mathcal{A}$: *associative algebra with co-product*
$\rightsquigarrow$ *tensor product of representations,*
$\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$: *Relates tensor products taken in different orders.*

and representations $\pi_\lambda^{\mathrm{aux}}$, π^{qu} of $\mathcal{A}$ such that

$$\boxed{\mathsf{L}(\lambda/\mu) = (\pi_\lambda^{\mathrm{aux}} \otimes \pi_\mu^{\mathrm{qu}})(\mathcal{R})}$$

Quantum affine algebra $\mathcal{A} = \mathcal{U}_q(\hat{\mathfrak{g}})$

Defining relations:

$$k_i e_j = q^{A_{ij}} e_j k_i, \qquad k_i f_j = q^{-A_{ij}} f_j k_i, \qquad e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$q^D e_i = q^{\delta_{i0}} e_i q^D, \qquad k_i k_j = k_j k_i, \qquad q^D k_i = k_i q^D, \qquad q^D f_i = q^{-\delta_{i0}} f_i q^D,$$

$$\sum_{n=0}^{1-A_{ij}} (-1)^n \left[{1-A_{ij} \atop n} \right]_q e_i^n e_j e_i^{1-A_{ij}-n} = \sum_{n=0}^{1-A_{ij}} (-1)^n \left[{1-A_{ij} \atop n} \right]_q f_i^n f_j f_i^{1-A_{ij}-n} = 0.$$

Borel subalgebras $\mathcal{B}_+$ / $\mathcal{B}_-$ generated by e_i, k_i / f_i, k_i, respectively.

Universal R-matrix $\mathcal{R}$:

$$\mathcal{R}\,\Delta(x) = \Delta^{\mathrm{op}}(x)\mathcal{R} \qquad \text{for all } x \in \mathcal{U}_q(\hat{\mathfrak{g}}),$$

$$(\Delta \otimes \mathbb{1})(\mathcal{R}) = \mathcal{R}_{13}\,\mathcal{R}_{23} \qquad \text{and} \qquad (\mathbb{1} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\,\mathcal{R}_{12}.$$

There exist **two** solutions $\mathcal{R}^+$ and $\mathcal{R}^- = (\sigma(\mathcal{R}^+))^{-1}$ where

$$\mathcal{R}^+ \in \mathcal{B}_+ \otimes \mathcal{B}_- \quad \text{and} \quad \mathcal{R}^- \in \mathcal{B}_- \otimes \mathcal{B}_+.$$

Note that $\mathcal{R}^+ = q^t \bar{\mathcal{R}}^+$ where $\bar{\mathcal{R}}^+ = 1 \otimes 1 + (q - q^{-1}) \sum_i e_i \otimes f_i + \ldots$.

Theory of q-integrable models ⊂ quantum group theory II

Example: (Higher spin) XXZ. Take evaluation representations of $\mathcal{A}$: Let $\mathcal{U}_q(\mathfrak{sl}_2)$ be the algebra generated by E, F and $\mathsf{K}^{\pm 1}$ with relations

$$\mathsf{KE} = q^{+1}\mathsf{EK}\,, \qquad \mathsf{KF} = q^{-1}\mathsf{FK}\,, \qquad [\,\mathsf{E}\,,\,\mathsf{F}\,] = \frac{\mathsf{K}^2 - \mathsf{K}^{-2}}{q - q^{-1}}.$$

"Upgrade" to a representation of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$ via

$$\mathrm{ev}_\lambda(e_1) = \lambda^{-1}\, q^{\frac{1}{2}}\, \mathsf{K}^{-1}\mathsf{E}\,, \quad \mathrm{ev}_\lambda(e_0) = \lambda^{-1}\, q^{\frac{1}{2}}\, \mathsf{K}^{+1}\,\mathsf{F}\,, \quad \mathrm{ev}_\lambda(k_1) = \mathsf{K}^{+2}\,,$$
$$\mathrm{ev}_\lambda(f_1) = \lambda^{+1}\, q^{\frac{1}{2}}\, \mathsf{K}^{+1}\,\mathsf{F}\,, \quad \mathrm{ev}_\lambda(f_0) = \lambda^{+1}\, q^{\frac{1}{2}}\, \mathsf{K}^{-1}\mathsf{E}\,, \quad \mathrm{ev}_\lambda(k_0) = \mathsf{K}^{-2}\,.$$

For higher spin XXZ use fin.-dim. representations π^j of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $\mathbb{C}^{2j+1}$

$$\mathsf{L}(\lambda/\mu) = (\pi_\lambda^{1/2} \otimes \mathrm{ev}_\mu)(\mathcal{R}^-)$$
$$\propto \begin{pmatrix} \mathsf{K}^{-1} - \lambda^2\mu^{-2}\, q\, \mathsf{K}^{+1} & \lambda\mu^{-1}(q^{-1} - q)\, q^{+\frac{1}{2}}\, \mathsf{K}^{+1}\mathsf{F}\mathsf{K}^{-1} \\ \lambda\mu^{-1}(q^{-1} - q)\, q^{+\frac{1}{2}}\, \mathsf{K}^{-1}\mathsf{E}\mathsf{K}^{-1} & \mathsf{K}^{+1} - \lambda^2\mu^{-2}\, q\, \mathsf{K}^{-1} \end{pmatrix}$$

Strength: Functional relations for eigenvalues from representation theory

(Bazhanov-Lukyanov-Zamolodchikov ... Frenkel-Hernandez)

Let **Osc** be a representation of the q-oscillator algebra

$$q^D \mathrm{a} q^{-D} = q^{-1}\mathrm{a}, \quad q^D \mathrm{a}^* q^{-D} = \mathrm{a}^* \quad \mathrm{a}\mathrm{a}^* = 1 - q^{2D+2}, \quad \mathrm{a}^*\mathrm{a} = 1 - q^{2D}.$$

Allows to define representation of **Borel subalgebra** $\mathcal{B}_+$ via

$$\pi_\lambda^{\mathrm{os}}(e_0) = \frac{\lambda}{q - q^{-1}}\mathrm{a}, \quad \pi_\lambda^{\mathrm{os}}(e_1) = \frac{\lambda}{q - q^{-1}}\mathrm{a}^*, \quad \begin{array}{l} \pi_\lambda^{\mathrm{os}}(k_0) = q^{-D}, \\ \pi_\lambda^{\mathrm{os}}(k_1) = q^{+D}. \end{array}$$

Q-operators defined as

$$\mathrm{Q}(\lambda/\mu) = \mathrm{Tr}_{\mathbf{Osc}}\left[(\pi_\lambda^{\mathrm{os}} \otimes \pi_\mu^j)(\mathcal{R}^+)\right]$$

Bethe Ansatz equations follow from Baxter equation

$$T(q^{\frac{1}{2}}\lambda)Q(\lambda) = Q(q\lambda) + Q(q^{-1}\lambda),$$

following from reducibility of $\pi_\lambda^{\mathrm{os}} \otimes \pi_\mu^{\frac{1}{2}}$ for $\mu = q^{\frac{1}{2}}\lambda$.

Problem: Generalisation to relativistic integrable QFT

Examples

Motivation

- Interesting QFT models, e.g. **affine Toda**

$$S=\int d^2z\left(\frac{1}{4\pi}\left((\partial_\alpha\phi_1)^2+(\partial_\alpha\phi_2)^2\right)+2\mu e^{-b\phi_1}\cosh(\sqrt{3}b\phi_2)+\nu e^{2b\phi_1}\right)$$

  or $\mathcal{N}=2$ **SUSY Sine-Gordon**

$$S=\int d^2z\,\Big(\frac{1}{4\pi}\left((\partial_\alpha\phi_1)^2+(\partial_\alpha\phi_2)^2\right)+\frac{1}{2\pi}\left(\bar\psi_+\partial_-\psi_++\bar\psi_-\partial_+\psi_-\right)$$
$$-2b\mu\left(\bar\psi_+\bar\psi_-\cosh(b(\phi_1+i\phi_2))+\psi_+\psi_-\cosh(b(\phi_1-i\phi_2))\right)$$
$$+4\pi\left(\mu^2e^{2b\phi_1}+\nu^2e^{-2b\phi_1}-2\mu\nu\cos(2b\phi_2)\right)\Big)$$

- Sometimes dual to sigma models (like $\mathcal{N}=2$ **SUSY sausage**)

Affine Toda – Lightcone Representation

$$\partial_+\partial_- \phi_i = -\frac{\pi}{2} b\mu\left(e^{2b(\phi_i-\phi_{i+1})} - e^{2b(\phi_{i-1}-\phi_i)}\right)$$

$$\Leftrightarrow \quad [\,\partial_+ - A_+(\lambda)\,,\,\partial_- - A_-(\lambda)\,] = 0$$

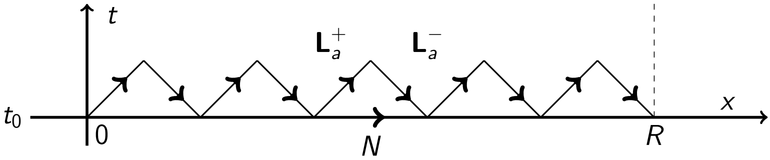
where $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$, and

$$A_+(\lambda) = \sum_{i=1}^{M}\left(-b(\partial_+\phi_i)\mathrm{E}_{ii} + m e^{b(\phi_i-\phi_{i+1})}\mathrm{E}_{i,i+1}\right)$$

$$A_-(\lambda) = \sum_{i=1}^{M}\left(+b(\partial_-\phi_i)\mathrm{E}_{ii} - m e^{b(\phi_i-\phi_{i+1})}\mathrm{E}_{i+1,i}\right)$$

Lightcone lattice discretisation

(Faddeev-Volkov .... Kashaev-Reshetikhin)



$$\mathcal{L}_a(\lambda) = \mathsf{L}^-_{2a}(q^{\frac{1}{2}}\kappa^{+1}\lambda)\,\mathsf{L}^+_{2a-1}(q^{\frac{1}{2}}\kappa^{-1}\lambda)$$

$$\mathsf{L}^-_r(\lambda) = \left(1 - q^{-1}\lambda^M\right)\left[\sum_{i=1}^{M}\left(\mathsf{u}_{i,r}^{-1}\,\mathsf{E}_{ii} - q^{-1}\,\lambda\,\mathsf{v}_{i,r}\,\mathsf{E}_{i+1i}\right)\right]^{-1}$$

$$\mathsf{L}^+_r(\lambda) = \sum_{i=1}^{M}\left(\mathsf{u}_{i,r}^{+1}\,\mathsf{E}_{ii} + \lambda^{-1}\,\mathsf{v}_{i,r}\,\mathsf{E}_{ii+1}\right)$$

$$\mathsf{u}_{i,r} = e^{-2\pi b\,\Pi_i(r)} \qquad [\Pi_i(r),\phi_j(s)] = (2\pi i)^{-1}\delta_{ij}\delta(r-s)$$

$$\mathsf{v}_{i,r} = e^{\pi b\,(\phi_i(r)-\phi_{i+1}(r))}$$

$\kappa \sim$ mass parameter.

Lightcone lattice from quantum groups I - Kinematics

From universal $\mathcal{R}$ using **"prefundamental"** representations:

$$
\begin{aligned}
\mathsf{L}^{+}(\lambda\mu^{-1}) &= \frac{1}{\theta^{+}(\lambda\mu^{-1})}\,\Big[\big(\pi^{\mathrm{f}}_{\lambda}\otimes\pi^{+}_{\mu}\big)\,\big(\mathcal{R}^{+}\big)\Big]_{\mathsf{ren}}\\
\mathsf{L}^{-}(\lambda\mu^{-1}) &= \frac{1}{\theta^{-}(\lambda\mu^{-1})}\,\Big[\big(\pi^{\mathrm{f}}_{\lambda}\otimes\pi^{-}_{\mu}\big)\,\big(\mathcal{R}^{-}\big)\Big]_{\mathsf{ren}}
\end{aligned}
$$

The relevant representations are defined as follows

$$
\pi^{\mathrm{f}}_{\lambda}(e_i) = \lambda^{-1}\,\mathsf{E}_{i,i+1}\,,\quad \pi^{\mathrm{f}}_{\lambda}(f_i) = \lambda\,\mathsf{E}_{i+1,i}\,,\quad \pi^{\mathrm{f}}_{\lambda}(h_i) = \mathsf{E}_{i,i}-\mathsf{E}_{i+1,i+1}\,,
$$

where E_{ij} are the matrix units $\mathsf{E}_{ij}\,\mathsf{E}_{kl} = \delta_{jk}\,\mathsf{E}_{il}$ and

$$
\begin{aligned}
\pi^{+}_{\lambda}(f_i) &= \frac{\lambda}{q-q^{-1}}\,\mathsf{u}_i^{-1}\,\mathsf{v}_i\,,\quad \pi^{+}_{\lambda}(k_i) = \mathsf{u}_i\,\mathsf{u}_{i+1}^{-1}\,,\\
\pi^{-}_{\lambda}(e_i) &= \frac{\lambda^{-1}}{q^{-1}-q}\,\mathsf{v}_i\,\mathsf{u}_{i+1}\,,\quad \pi^{-}_{\lambda}(k_i) = \mathsf{u}_i^{-1}\,\mathsf{u}_{i+1}\,.
\end{aligned}
\qquad \{\mathsf{v}_i,\mathsf{u}_i\}_{i=1,\dots,M}.
$$

$\theta^{+}(x)$, $\theta^{-}(x)$ are certain scalar factors.

Lightcone lattice from quantum groups II - Dynamics

Lightcone evolution

$$\mathsf{O}_{r+1,\tau+1} := (\mathsf{U}_\kappa^+)^{-1} \cdot \mathsf{O}_{r,\tau} \cdot \mathsf{U}_\kappa^+ , \quad \mathsf{O}_{r-1,\tau+1} := (\mathsf{U}_\kappa^-)^{-1} \cdot \mathsf{O}_{r,\tau} \cdot \mathsf{U}_\kappa^- .$$

generated by

$$\mathsf{U}_\kappa^+ = \left[\prod_{a=1}^{N} \mathsf{r}_{2a,2a-1}^{-+}(\bar{\mu},\mu) \right] \cdot \mathsf{C}_{\rm odd} , \quad \mathsf{U}_\kappa^- = \left[\prod_{a=1}^{N} \mathsf{r}_{2a,2a-1}^{-+}(\bar{\mu},\mu) \right] \cdot \mathsf{C}_{\rm even}^{-1} .$$

where $\kappa^2 = \mu^{-1}\bar{\mu}$. The operators $\mathsf{C}_{\rm odd}$ and $\mathsf{C}_{\rm even}$ are defined such that

$$\begin{aligned} \mathsf{C}_{\rm odd} \cdot \mathsf{O}_{2a+1} &= \mathsf{O}_{2a-1} \cdot \mathsf{C}_{\rm odd} , & \mathsf{C}_{\rm odd} \cdot \mathsf{O}_{2a} &= \mathsf{O}_{2a} \cdot \mathsf{C}_{\rm odd} , \\ \mathsf{C}_{\rm even} \cdot \mathsf{O}_{2a-1} &= \mathsf{O}_{2a-1} \cdot \mathsf{C}_{\rm even} , & \mathsf{C}_{\rm even} \cdot \mathsf{O}_{2a} &= \mathsf{O}_{2a-2} \cdot \mathsf{C}_{\rm even} , \end{aligned}$$

We then have

$$\boxed{\mathsf{r}^{-+}(\lambda/\mu) = \frac{1}{\rho^{-+}(\lambda\mu^{-1})} \left[\left(\pi_\lambda^- \otimes \pi_\mu^+\right)(\mathcal{R}) \right]_{\sf ren}}$$

Q-operators – I

Look for fundamental R-matrix $\mathcal{R}_{AB}$,

$$\mathcal{L}_A(\bar{\mu},\mu)\mathcal{L}_B(\bar{\nu},\nu)\,\mathcal{R}_{AB}(\bar{\mu},\mu;\bar{\nu},\nu) = \mathcal{R}_{AB}(\bar{\mu},\mu;\bar{\nu},\nu)\mathcal{L}_B(\bar{\nu},\nu)\mathcal{L}_A(\bar{\mu},\mu)$$

Then get Q-operators as

$$\mathcal{Q}(\bar{\mu},\mu;\bar{\nu},\nu) = \mathrm{Tr}_{\mathcal{H}_0^-\otimes\mathcal{H}_0^+}\big(\,\mathcal{R}_{0N}(\bar{\mu},\mu;\bar{\nu},\nu)\,\ldots\,\mathcal{R}_{01}(\bar{\mu},\mu;\bar{\nu},\nu)\,\big)\,.$$

Thanks to factorisation $\mathcal{L}_A(\bar{\mu},\mu) = \mathsf{L}_{\bar{a}}^-(\bar{\mu})\mathsf{L}_a^+(\mu)$ can solve

$$\mathcal{R}_{AB}(\bar{\mu},\mu;\bar{\nu},\nu) \;=\; \mathsf{r}_{a,\bar{b}}^{+-}(\mu,\bar{\nu})\,\mathsf{r}_{a,b}^{++}(\mu,\nu)\,\mathsf{r}_{\bar{a},\bar{b}}^{--}(\bar{\mu},\bar{\nu})\,\mathsf{r}_{\bar{a},b}^{-+}(\bar{\mu},\nu)$$

where

$$\begin{aligned}
\mathsf{L}_m^+(\mu)\,\mathsf{L}_n^-(\nu)\,\mathsf{r}_{m,n}^{+-}(\mu,\nu) &= \mathsf{r}_{m,n}^{+-}(\mu,\nu)\,\mathsf{L}_n^-(\nu)\,\mathsf{L}_m^+(\mu)\,,\\
\mathsf{L}_m^+(\mu)\,\mathsf{L}_n^+(\nu)\,\mathsf{r}_{m,n}^{++}(\mu,\nu) &= \mathsf{r}_{m,n}^{++}(\mu,\nu)\,\mathsf{L}_n^+(\nu)\,\mathsf{L}_m^+(\mu)\,,\\
\mathsf{L}_m^-(\mu)\,\mathsf{L}_n^-(\nu)\,\mathsf{r}_{m,n}^{--}(\mu,\nu) &= \mathsf{r}_{m,n}^{--}(\mu,\nu)\,\mathsf{L}_n^-(\nu)\,\mathsf{L}_m^-(\mu)\,,\\
\mathsf{L}_m^-(\mu)\,\mathsf{L}_n^+(\nu)\,\mathsf{r}_{m,n}^{-+}(\mu,\nu) &= \mathsf{r}_{m,n}^{-+}(\mu,\nu)\,\mathsf{L}_n^+(\nu)\,\mathsf{L}_m^-(\mu)\,.
\end{aligned}$$

Q-operators -II

Missing building block is $\mathsf{r}^{++}(\lambda)$. Obtained from universal $\mathcal{R}$ as

$$\mathsf{r}^{++}(\lambda) \,=\, (1\otimes\mathcal{F})\cdot\bar{\mathsf{r}}^{+-}(\lambda)\cdot(\Omega\otimes\mathcal{F})^{-1}\,.$$

where

- $\mathcal{F}\cdot\mathsf{u}_i\cdot\mathcal{F}^{-1}=\mathsf{v}_i$, $\mathcal{F}\cdot\mathsf{v}_i\cdot\mathcal{F}^{-1}=\mathsf{u}_{i+1}$,
- $\Omega=\mathcal{F}^2$: generator of $\mathbb{Z}^N$ symmetry,
- $$\bar{\mathsf{r}}^{+-}(\lambda\,\mu^{-1}) \,=\, \frac{1}{\rho^{+-}(\lambda\mu^{-1})}\left[(\pi^+_\lambda\otimes\bar{\pi}^-_\mu)(\mathcal{R}^-)\right]_{\mathsf{ren}}\,,$$
  $$\bar{\pi}^-_\lambda(e_i) \,=\, \frac{\lambda^{-1}}{q^{-1}-q}\,\mathsf{v}_i\,\mathsf{u}_i^{-1}\,,\quad \bar{\pi}^-_\lambda(k_i) \,=\, \mathsf{u}_i^{-1}\,\mathsf{u}_{i+1}\,.$$

Note: $\bar{\pi}^-$ is the **conjugate** to π^- in the sense that $\bar{L}^-\sim(L^-)^{-1}$.

Meaning of “renormalisation” ???

Quantum affine algebras II - Product formula

$$R^- \;=\; \bar{\mathcal{R}}^- \, q^{-t} \;=\; \mathcal{R}^-_{\prec\delta}\, \mathcal{R}^-_{\sim\delta}\, \mathcal{R}^-_{\succ\delta}\, q^{-t}\,.$$

$\bar{\mathcal{R}}^-$ is an infinite ordered product over positive roots, factors

$$\mathcal{R}^-_\gamma \;=\; \exp_{q^{(\gamma,\gamma)}}\left((q^{-1}-q)\, s_\gamma^{-1}\, f_\gamma \otimes e_\gamma\right) \qquad \gamma \in \Delta_+^{\mathrm{re}}(\widehat{\mathfrak{g}})\,,$$

with $\exp_q(x)$ the quantum exponential

$$\exp_q(x) \;=\; \sum_{n=0}^{\infty} \frac{1}{(n)_q!} x^n\,, \quad (k)_q = \frac{q^k-1}{q-1}\,, \quad (n)_q! = (1)_q\,(2)_q\ldots(n)_q\,.$$

The contribution of positive imaginary roots is given by

$$\mathcal{R}^-_{\sim\delta} = \exp\left((q^{-1}-q^{+1}) \sum_{m\in\mathbb{Z}_+} \sum_{i,j=1}^{r} u_{m,\,ij}\, f^{(i)}_{m\delta} \otimes e^{(j)}_{m\delta}\right),$$

where r is the rank of the Lie algebra $\mathfrak{g}$, and

$$u_{m,ij} \;=\; \frac{m}{[M\,m]_q}\, [M-\max(i,j)]_{q^m}\, [\min(i,j)]_{q^m}\, (-1)^{m(i-j)}\,,$$

Why the product formula is useful – 1

Truncates to **finite** products for "prefundamental" representations as most positive (negative) root generators get represented by zero.

Why the product formula is useful – 2

Special functions appearing in it have natural replacements which are self-dual under $b \to b^{-1}$, may e.g. replace

$$\varepsilon_q(w) = \exp_{q^2}((q - q^{-1})^{-1} w) \;\text{ by }\; \mathcal{E}_{b^2}(w) = \exp\left(\Theta_{b^2}\left(\tfrac{1}{2\pi b}\log w\right)\right)$$

$$\Theta_{b^2}(x) := \int_{\mathbb{R}+i0} \frac{dt}{4t}\,\frac{e^{-2itx}}{\sinh(bt)\sinh(t/b)}$$

In our representations $w \to \mathbf{w}$, $\mathbf{w}$: positive self-adjoint operator. Magic of modular duality!

Modular double of quantum affine algebras...? – !

Yes, but renormalisation of imaginary root contributions subtle....

$\rightsquigarrow$ **"more universal R-matrix" !**

Towards the solution - I

Two main pieces of information needed:

- Functional relations - from algebra, e.g.

$$\sum_{k=0}^{M} (-1)^k \, \mathbb{T}^{(k)}(q^{\frac{k}{M}}\zeta)\, \mathrm{Q}^{+}(-\omega\, q^{\frac{2k-M}{M}}\zeta) \;=\; 0\,,$$

- Analytic properties - from kernel of Q-operator, e.g. for $M = 2$

$$\left[\begin{array}{ll} (\mathrm{Q}_1) & q(u) \sim e^{-\mathrm{i}\frac{\pi}{2}(u\mp(s+i\delta))^2 N} \quad \text{for} \;\; |u| \to \infty, \quad |\arg(\pm u)| < \frac{\pi}{2}\,, \\ (\mathrm{Q}_2) & q(u) \text{ is meromorphic with poles of maximal order } N \\ & \text{in } \;\pm\Upsilon_{-s}, \;\text{ the poles at } s \pm u = i\delta \text{ have order } N, \end{array}\right]$$

(Bytsko, J.T. 2006)

Result: quantisation conditions,

formulated via TBA-type NLIE ($M = 2$: hep-th/0702214, $M > 2$: to be done)

Towards the solution –II

$$Q_{\bar{\mu},\mu;\bar{\nu},\nu}(y,y') = \int d\mu_N(x)\ \prod_{a=1}^{N} \check{R}_{\bar{\mu},\mu;\bar{\nu},\nu}(x_{a+1},y_a|x'_a,y'_a)$$

$$\check{R}_{\bar{\mu},\mu;\bar{\nu},\nu}(x,y|x',y') := \langle\, x,y\,|\,\check{\mathsf{R}}_{AB}(\bar{\mu},\mu;\bar{\nu},\nu)\,|\,x',y'\,\rangle\,,$$

$$= \delta(\bar{x}-\bar{x}')\delta(\bar{y}-\bar{y}')\,W^{+-}_{\bar{\nu}/\mu}(x,y)\,W^{++}_{\nu/\mu}(x,x')\,W^{--}_{\bar{\nu}/\bar{\mu}}(y,y')\,W^{-+}_{\nu/\bar{\mu}}(x',y')\,,$$

using the notations $\bar{x} = \sum_{i=1}^{M} x_i$ for $x \in \mathbb{R}^M$,

$$W^{++}_{\lambda}(x,x') = W^{--}_{\lambda}(x',x) = e^{\pi\mathrm{i}P(x,x')}\,\bar{V}_w(x-x')\,,$$

$$W^{-+}_{\lambda}(x,y) = \big(W^{+-}_{1/\lambda}(x,y)\big)^{-1} = e^{\pi\mathrm{i}P(x,y)}\,V_w(x-y)\,;$$

$P(x,y) = \sum_{i=1}^{M}(x_i y_{i+1} - y_i x_{i+1})$, $w = \frac{1}{2\pi b}\log\lambda$,

$$V_w(s) = \prod_{i=1}^{M} \frac{e^{-\frac{\pi\mathrm{i}}{2}w^2}}{\mathbf{s}_b(s_{i,i+1}+w)}\,, \qquad \bar{V}_w(s) = N_w \prod_{i=1}^{M} \mathbf{s}_b(w - s_{i,i+1} + c_b)\,.$$

The resulting expression resembles the one found for the generalised chiral Potts models.

Summary

- Systematic approach to lightcone lattice discretisation from quantum group theory
- “Prefundamental” representations: Building blocks for all relevant objects
- Product formula admits renormalisation for certain classes of infinite-dimensional representations
- Main ingredients for exact solution are obtained in this way.

Research is often like finding a way through the jungle,

one may stumble upon scary obstacles

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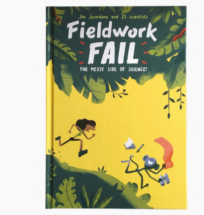

Many of us would turn around,

Jean-Michel would fight the caiman!

Happy birthday, Jean-Michel!