qKZ equations and current fluctuations in the open ASEP

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with C. FINN.

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Fluctuations of the current in open ASEP

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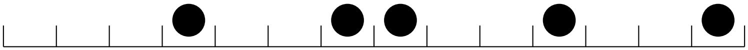

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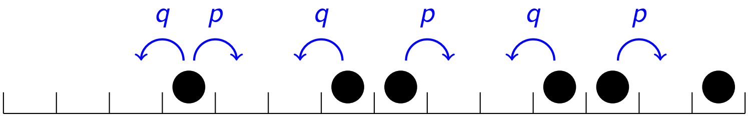

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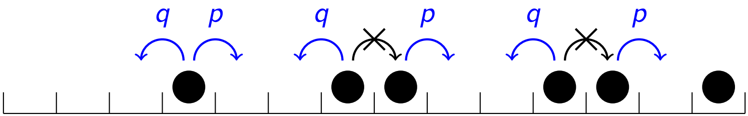

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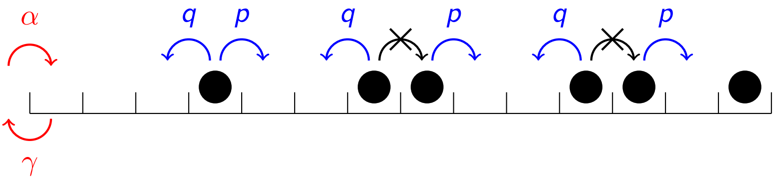

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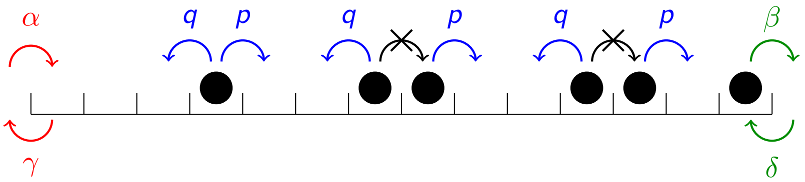

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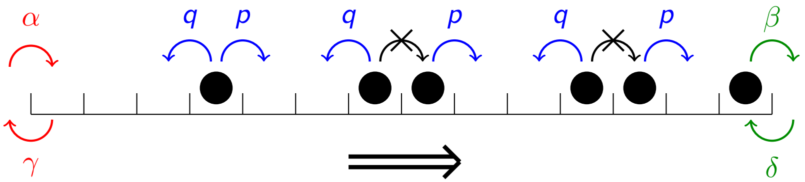

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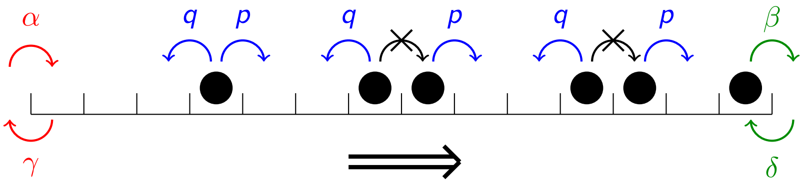

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$\longrightarrow$ **Goal: study the statistics of this current**

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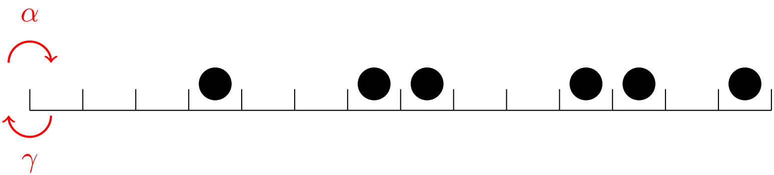

Markov matrix:

$$M = B_1 + \sum_{k=1}^{L-1} m_{k,k+1} + \overline{B}_L,$$

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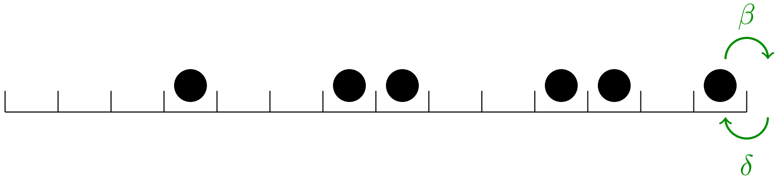

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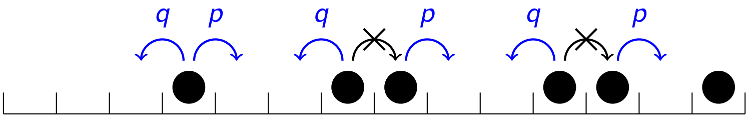

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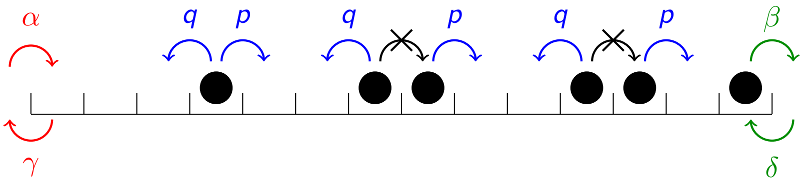

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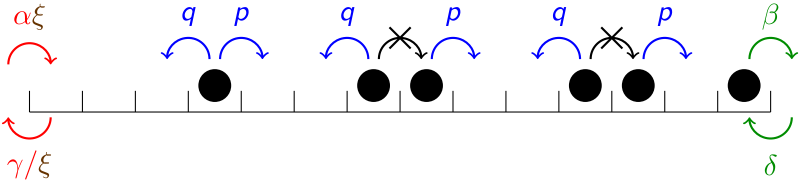

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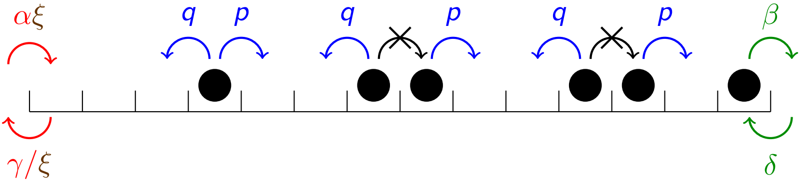

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Deformation ξ “counts” injected/removed particles

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$\longrightarrow$ **Goal: compute this leading eigenvalue.** (Lazarescu, Mallick)

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$$\check{R}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{p-q}{p-qx} & \frac{p(1-x)}{p-qx} & 0 \\ 0 & \frac{q(1-x)}{p-qx} & \frac{x(p-q)}{p-qx} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Yang-Baxter equation

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- It ensures the integrability of the bulk dynamics

$$(q-p)\check{R}'(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = m$$

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$$K(x)=\begin{pmatrix} \frac{x^2(\gamma-\alpha)+x(\alpha-\gamma+q-p)}{x^2\gamma+x(\alpha-\gamma+q-p)-\alpha} & \frac{(x^2-1)\gamma/\xi}{x^2\gamma+x(\alpha-\gamma+q-p)-\alpha} \\ \frac{(x^2-1)\alpha\xi}{x^2\gamma+x(\alpha-\gamma+q-p)-\alpha} & \frac{\gamma-\alpha+x(\alpha-\gamma+q-p)}{x^2\gamma+x(\alpha-\gamma+q-p)-\alpha} \end{pmatrix}$$

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Reflection equations

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with $K_1(x) = K(x)\otimes 1$, $K_2(x) = 1\otimes K(x)$, ...

- It ensures the integrability of the boundary conditions

$$\frac{(q-p)}{2}K'(1) = \begin{pmatrix} -\alpha & \gamma/\xi \\ \alpha\xi & -\gamma \end{pmatrix} = B(\xi), \qquad \frac{(p-q)}{2}\overline{K}'(1) = \begin{pmatrix} -\delta & \beta \\ \delta & -\beta \end{pmatrix} = \overline{B}$$

- They are the building blocks of the transfer matrix (Sklyanin)

$$t(x;\mathbf{x}) = tr_0\Big(\widetilde{K}_0(x)\, R_{0L}\left(\frac{x}{x_L}\right)\dots R_{01}\left(\frac{x}{x_1}\right)\, K_0(x)\, R_{10}(x\,x_1)\dots R_{L0}(x\,x_L)\Big),$$

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- In fact the deformed Markov matrix is related to the XXZ spin chain with integrable boundaries

$$M(\xi) = U H_{XXZ} U^{-1}$$

qKZ equations and matrix ansatz

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$$M(\xi)|\Psi(\xi)\rangle = \Lambda_0(\xi)|\Psi(\xi)\rangle.$$

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- We want to solve the exchange relations

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$$K_1\,(1/x_1)|\Psi(1/x_1, x_2 \dots)\rangle = |\Psi(sx_1, x_2 \dots)\rangle,$$
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- These relations are called qKZ equations.

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$$\psi_{\boldsymbol{\tau}}(\mathbf{x}; s, \xi) = \langle\langle W|\mathbb{S}A_{\tau_1}(x_1)\dots A_{\tau_{L-1}}(x_{L-1})A_{\tau_L}(x_L)|V\rangle\rangle.$$

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- The deformed ground state is concisely written

$$|\Psi(\mathbf{x}; s, \xi)\rangle = \langle\langle W|\mathbb{S}\mathbb{A}(x_1) \otimes \dots \otimes \mathbb{A}(x_L)|V\rangle\rangle,$$

$$\text{with} \qquad \mathbb{A}(x) = \begin{pmatrix} A_0(x) \\ A_1(x) \end{pmatrix}.$$

- In this context the qKZ equation translates into:

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- There exists an explicit polynomial solution $|\Psi^{(m)}(\mathbf{x};s)\rangle$ when

$$\xi=s^m,\qquad m\geq 1$$

Explicit construction of the solution

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The building blocks are a deformed oscillator algebra generated by **e**, **d** and **S**

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$$p\mathbf{de} - q\mathbf{ed} = p - q, \quad \mathbf{dS} = s\mathbf{Sd}, \quad \mathbf{Se} = s\mathbf{eS}$$

and paired boundary vectors $\langle\langle w|$ and $|v\rangle\rangle$

$$\langle\langle w| \,(\alpha\mathbf{e} - \gamma\mathbf{d}) = \langle\langle w| \,(p - q + \gamma - \alpha)\,,$$
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and $\langle\langle \widetilde{w}|$ and $|\widetilde{v}\rangle\rangle$:

$$\langle\langle \widetilde{w}| (\alpha\mathbf{e} - \gamma\mathbf{d}) = \langle\langle \widetilde{w}| (\alpha - \gamma),$$
$$(\beta\mathbf{d} - \delta\mathbf{e}) |\widetilde{v}\rangle\rangle = (\beta - \delta) |\widetilde{v}\rangle\rangle.$$

Building on this algebra, we define

$$\mathbb{S}^{(m)} = \mathbf{S}^{2m-1} \otimes \mathbf{S}^{2m-2} \otimes \ldots \otimes \mathbf{S}^{3} \otimes \mathbf{S}^{2} \otimes \mathbf{S},$$
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We also define boundary vectors

$$\langle\langle W^{(m)}| = \underbrace{\langle\langle w| \otimes \langle\langle \widetilde{w}| \otimes \ldots \otimes \langle\langle w| \otimes \langle\langle \widetilde{w}|}_{m-1 \text{ times}} \otimes \langle\langle w|$$

$$|V^{(m)}\rangle\rangle = \underbrace{|v\rangle\rangle \otimes |\widetilde{v}\rangle\rangle \otimes \ldots \otimes |v\rangle\rangle \otimes |\widetilde{v}\rangle\rangle}_{m-1 \text{ times}} \otimes |v\rangle\rangle.$$

Main result

For integer $m > 0$ and $\xi = s^m$,

$$|\Psi^{(m)}(\mathbf{x}; s)\rangle = \frac{1}{\Omega^{(m)}} \langle\langle W^{(m)}|\mathbb{S}^{(m)}\mathbb{A}^{(m)}(x_1) \otimes \ldots \otimes \mathbb{A}^{(m)}(x_N)|V^{(m)}\rangle\rangle,$$

with normalisation factor

$$\Omega^{(m)} = \langle\langle W^{(m)}|\mathbb{S}^{(m)}|V^{(m)}\rangle\rangle,$$

is a solution of the qKZ equations.

Koornwinder polynomials and conjecture

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- We can show that $\mathcal{Z}^{(m)}(\mathbf{x}; s)$ is the symmetric Koornwinder polynomial $P_{(m)^L}(\mathbf{x})$.
- Symmetric Koornwinder polynomials are a family of orthogonal polynomials: they satisfy difference equations and they have a contour integral expression.

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$$\mathcal{S}_i(\mathbf{x})|_{s=1} = t(x_i|\mathbf{x}), \qquad \frac{\partial}{\partial x_i}\mathcal{S}_i(\mathbf{x})|_{s=x_1=\ldots=x_N=1} = \frac{2}{p-q}M(\xi).$$

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- For $s \to 1$, the scattering relation seems to degenerate into an eigenvalue equation

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We conjecture that

$$\lim_{m\to\infty} \frac{|\Psi^{(m)}(\mathbf{x}; s = \xi^{1/m})\rangle}{\mathcal{Z}^{(m)}(\mathbf{x}; s = \xi^{1/m})} = |\Psi_0(\mathbf{x}; \xi)\rangle,$$

$$\lim_{m\to\infty} \frac{\ln(\xi)}{m} \ln\left(\mathcal{Z}^{(m)}(\mathbf{x}; s = \xi^{1/m})\right) = F_0(\mathbf{x}; \xi),$$

where $|\Psi_0\rangle$ and F_0 are smooth functions of $\mathbf{x}$.

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where $|\Psi_0\rangle$ and F_0 are smooth functions of $\mathbf{x}$.
The second part of the conjecture can be rewritten with symmetric Koornwinder

$$\lim_{m\to\infty} \frac{\ln(\xi)}{m} \ln\left(P_{(m^L)}(\mathbf{x}; s = \xi^{1/m})\right) = F_0(\mathbf{x}; \xi).$$

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- We thus get a formula for the cumulants generating function of the current

$$E(\mu) = \Lambda_0(e^{\mu}) = \frac{p-q}{2}\frac{\partial^2 F_0}{\partial x_i^2}(\mathbf{1};e^{\mu}).$$

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- It translates into a differential equation for F_0:

$$\sum_{i=1}^{L} g_i(\mathbf{x}) \left[\exp\left(x_i \frac{\partial F_0}{\partial x_i}(\mathbf{x};\xi)\right) - 1\right] + \sum_{i=1}^{L} g_i(\mathbf{x}^{-1}) \left[\exp\left(-x_i \frac{\partial F_0}{\partial x_i}(\mathbf{x};\xi)\right) - 1\right]$$

$$= \frac{1-\left(\frac{p}{q}\right)^L}{1-\frac{p}{q}}(\xi-1)\left(\frac{\alpha}{\gamma}\frac{\beta}{\delta}\left(\frac{p}{q}\right)^{L-1} - 1/\xi\right)$$

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$$g_i(\mathbf{x}) = \frac{(\gamma-(p-q+\gamma-\alpha)x_i-\alpha x_i^2)(\delta-(p-q+\delta-\beta)x_i-\beta x_i^2)}{\gamma\delta(1-x_i^2)^2} \times \prod_{\substack{j=1\\ j\neq i}}^{L} \frac{(qx_j-px_i)(q-px_ix_j)}{q^2(x_j-x_i)(1-x_ix_j)},$$

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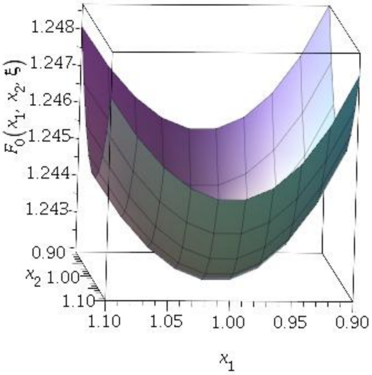

Thank you!