Exact non-equilibrium dynamics of quantum integrable models

Eric Vernier (Oxford)

joint work with **L. Piroli** (SISSA Trieste) and **B. Pozsgay** (Budapest)

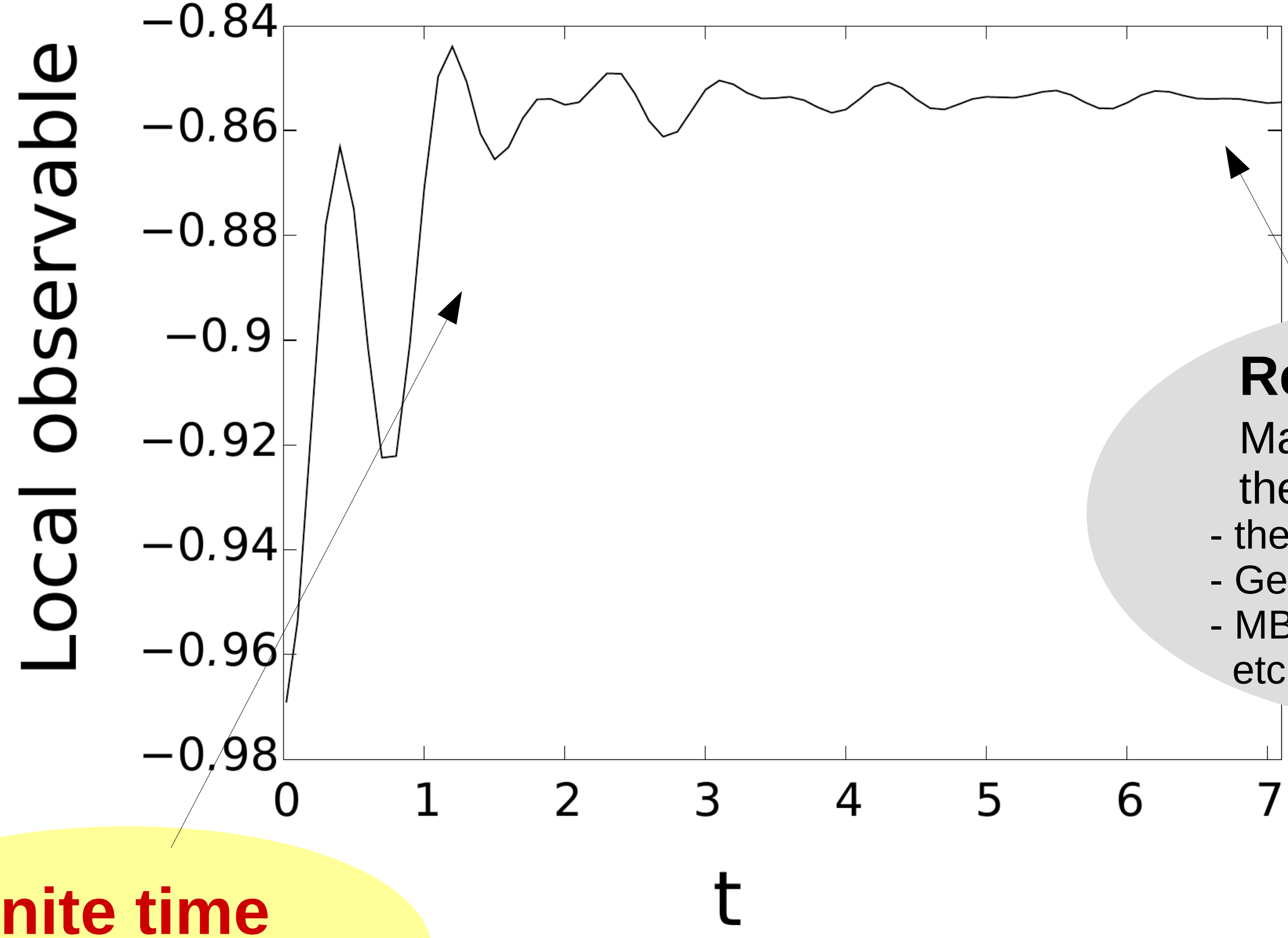
arXiv:1611.06126
arXiv:1709.04796
+ work in preparation

JMM60 Conference
Lyon, October 2017

Dynamics of isolated quantum systems

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

“quantum quench”



Relaxation

Many works over the last 10 years

- thermalization
- Generalized Gibbs Ensembles
- MBL

etc...

Finite time dynamics ?

Why studying finite time dynamics is important :

- inherent limitation of cold atomic experiments

- rich physics :

  prethermalization
  dynamical phase transitions **Heyl Polkovnikov Kehrein (2013)**
  particle-antiparticle production in lattice gauge theories **Blatt *et al* (2016)**
  etc….

Most results so far are numerical (iTEBD, DMRG): **time limitations !**

This talk : exact results using quantum integrability ?

very few exact results so far:

Non-interacting systems (free fermions, CFT, etc…)

“Dyson’s method” for integrable systems **D. Iyer, N. Andrei (2014)**
however, very limited use (eg: Bose gas with N=4 bosons)

Quench from a domain wall initial state in the XXZ chain, however “combinatorially special” in some sense **J.-M. Stéphan (2017); see his talk !**

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$$

$$\langle\mathcal{O}(t)\rangle = \sum_{i,j} \mathrm{e}^{i(E_i - E_j)t}\langle\Psi_0|i\rangle\langle i|\mathcal{O}|j\rangle\langle j|\Psi_0\rangle$$

sum over all eigenstates
+ overlaps

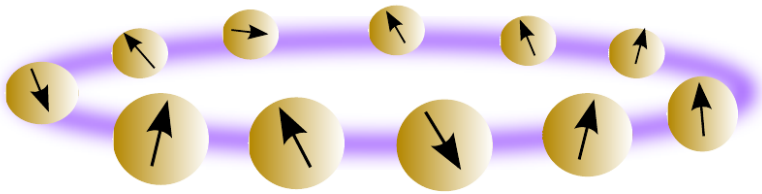
Form factors

1. Can we get exact results for some observables ?

2. For which initial states ? ("integrable initial states" ?)

Our starting point :

Model : Heisenberg XXZ chain

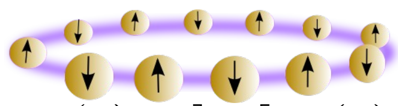


$$H = J \sum_{i=1}^{L} \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z$$

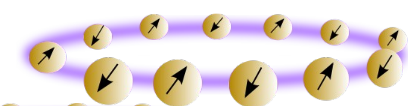
Initial state : 2-site product state

$$|\Psi_0\rangle = |\psi_0\rangle_{1,2} \otimes |\psi_0\rangle_{3,4} \cdots \otimes |\psi_0\rangle_{L-1,L} \qquad |\psi_0\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

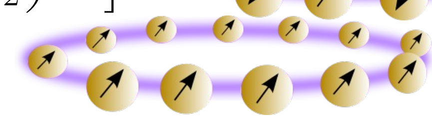
Néel $|\psi_0\rangle = |\uparrow\rangle \otimes |\downarrow\rangle$



Tilted Néel $|\psi_0\rangle = \left[-\sin\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + \cos\left(\frac{\vartheta}{2}\right)|\downarrow\rangle\right] \otimes \left[\cos\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\vartheta}{2}\right)|\downarrow\rangle\right]$



Tilted ferro $|\psi_0\rangle = \left[\cos\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\vartheta}{2}\right)|\downarrow\rangle\right] \otimes \left[\cos\left(\frac{\vartheta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\vartheta}{2}\right)|\downarrow\rangle\right]$



Observable : Loschmidt echo (“fidelity”, “return rate”)

$$\mathscr{L}(t) = \left|\langle\Psi_0|e^{-iHt}|\Psi_0\rangle\right|^2 \qquad \ell(t) = [\mathscr{L}(t)]^{1/L}$$ Loschmidt echo per site

- Relevant in many physical contexts : impurity problems, dynamical phase transitions

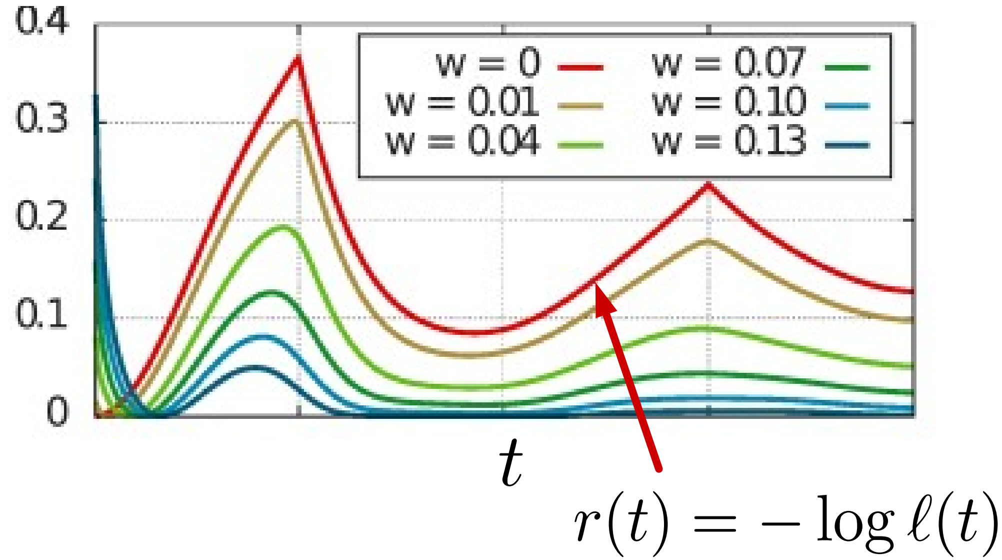
- Accessible experimentally (NMR: Fourier transform of the absorption spectrum)

Dynamical phase transitions

Heyl, Polkovnikov, Kehrein (2012)

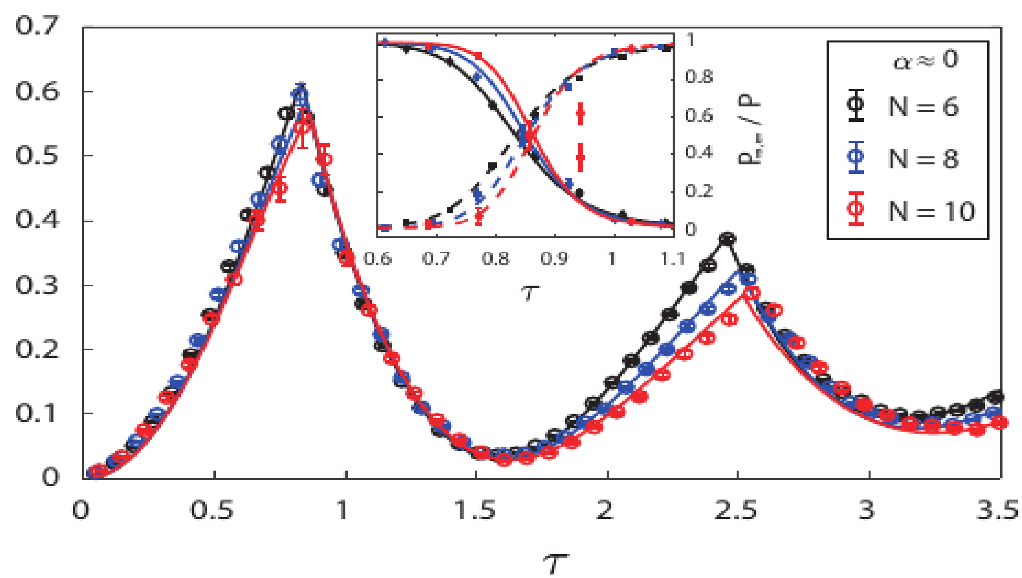
Quench across the critical point in the Ising chain

$$H = \sum_{j=1}^{L} \sigma_j^z \sigma_{j+1}^z - g\sigma_j^x \qquad g_i = 0.5 \to g_f = 2.0$$



Blatt et. al. (2016)

Experimental observation
on a chain of calcium-40 ions



Our strategy

original idea : **B. Pozsgay (2014)**
Very similar in spirit to the Quantum Transfer Matrix for thermal averages **Klümper (1992)**

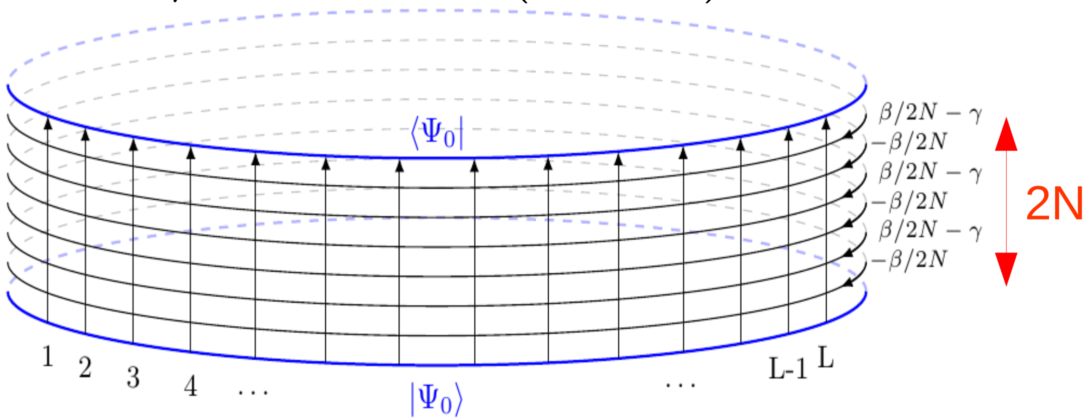
1. Express Loschmidt echo as a 2d classical parition function

$$\left\langle \Psi_0 \middle| e^{-wH} \middle| \Psi_0 \right\rangle = \lim_{N\to\infty} \langle \Psi_0 | \left(1 - \frac{wH}{N}\right)^N |\Psi_0\rangle \propto \lim_{N\to\infty} \langle \Psi_0 | \left[t\left(-\frac{\beta}{2N}\right) t\left(-\gamma + \frac{\beta}{2N}\right) \right]^N |\Psi_0\rangle$$

im. time

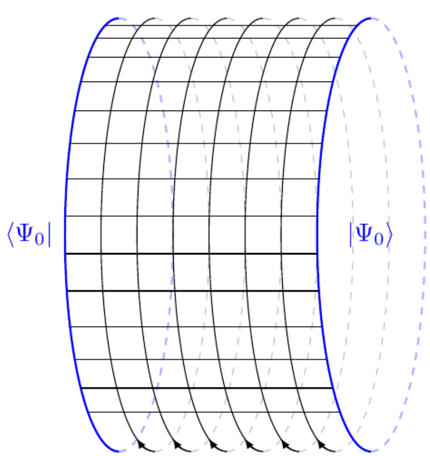
$$\beta = \frac{1}{2}\sin(\gamma) w$$

$$= \lim_{N\to\infty}$$

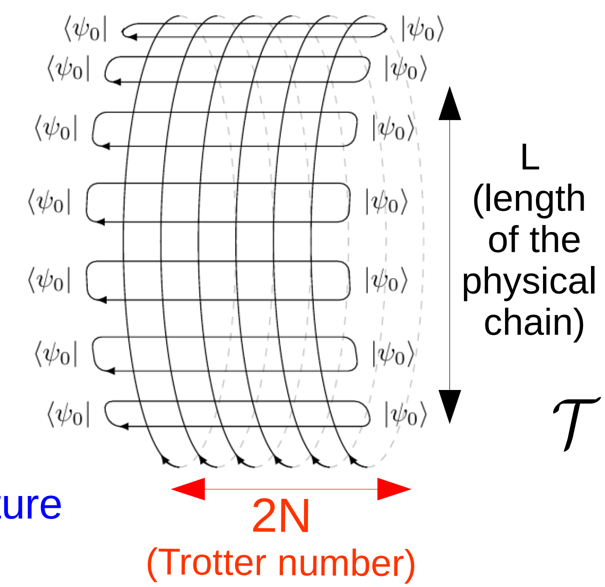


Partition function of an inhomogeneous six-vertex model with boundary conditions specified by $|\Psi_0\rangle$

2. Write as generated by a boundary Quantum Transfer Matrix in the transverse channel

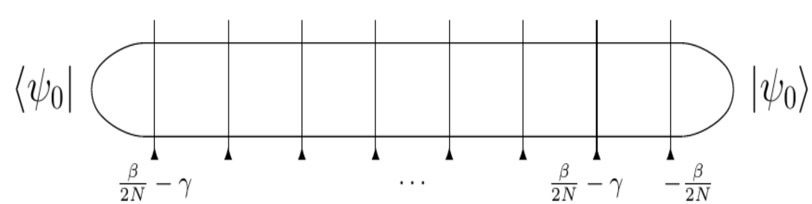


$$=$$



$$= \mathrm{tr}\left[\mathcal{T}^{L/2}\right]$$

$$\mathcal{T} =$$



Boundary QTM

Using the 2-site product structure

$$|\Psi_0\rangle = |\psi_0\rangle^{\otimes L/2}$$

Thermodynamic limit

$$\lim_{L\to\infty} \langle \Psi_0 | e^{-wH} | \Psi_0 \rangle^{1/L} = \lim_{L\to\infty} \left(\lim_{N\to\infty} \mathrm{tr}\left[\mathcal{T}^{L/2} \right] \right)^{1/L}$$

Usual QTM assumptions :

1. The two limits can be exchanged
2. $\mathcal{T}$ has a non-degenerate leading eigenvalue Λ

Then :

$$\lim_{L\to\infty} \langle \Psi_0 | e^{-wH} | \Psi_0 \rangle^{1/L} = \lim_{N\to\infty} \Lambda^{1/2}$$

Thermodynamic limit

$$\lim_{L\to\infty} \langle \Psi_0 | e^{-wH} | \Psi_0 \rangle^{1/L} = \lim_{L\to\infty} \left(\lim_{N\to\infty} \mathrm{tr}\left[\mathcal{T}^{L/2} \right] \right)^{1/L}$$

Usual QTM assumptions :

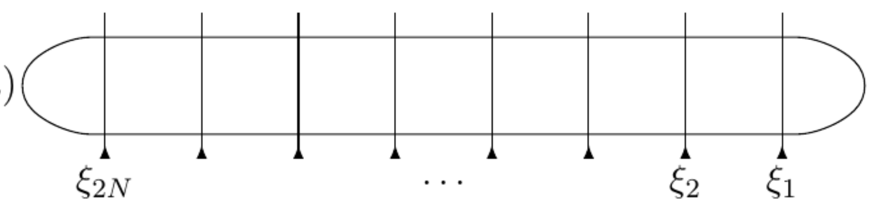
1. The two limits can be exchanged
2. $\mathcal{T}$ has a non-degenerate leading eigenvalue Λ

Then :

$$\lim_{L\to\infty} \langle \Psi_0 | e^{-wH} | \Psi_0 \rangle^{1/L} = \lim_{N\to\infty} \Lambda^{1/2}$$

Now a crucial fact :

any choice of $|\psi_0\rangle$ can be related to an **open integrable** transfer matrix

$T_{\mathrm{open}}(u) = K^+(u)$  $K^-(u)$

$$= \mathrm{tr}_0 \left[K^+(u) R_{0,2N}(u-\xi_{2N}) \ldots R_{0,1}(u-\xi_1) \right.$$
$$\left. K^-(u) R_{0,1}(u+\xi_1-\gamma) \ldots R_{0,2N}(u+\xi_{2N}-\gamma) \right]$$

$$\mathcal{T} = T_{\mathrm{open}}(0)$$

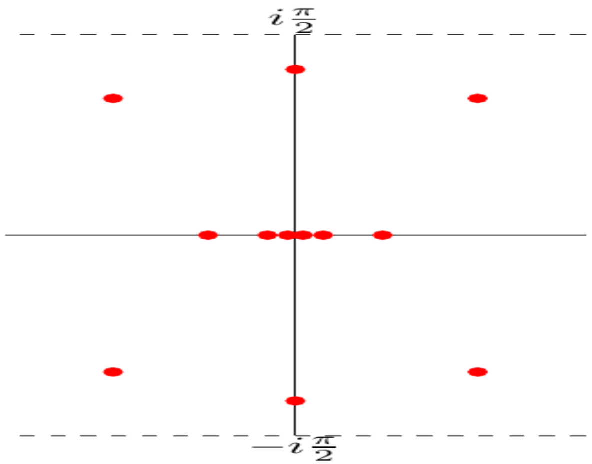
$K^{\pm}(u)$ solution of the reflection equation **Sklyanin (1988)**

So Λ can be calculated through "Bethe ansatz" !

Kitanine Maillet Niccoli Faldella
Wang Yang Cao Shi Lin
Frappat Nepomechie Ragoucy

Computing Λ

$\Lambda(u)$ can be written in terms of a set of $2N$ Bethe roots



$$\Lambda(u) = \mathbf{A}(u)\frac{Q(u-\gamma)}{Q(u)} + \mathbf{A}(-u)\frac{Q(u+\gamma)}{Q(u)} + \frac{F(u)}{Q(u)}$$

$$Q(u) = \prod_k \sin(u - i\lambda_k)\sin(u + i\lambda_k)$$

However they are difficult to handle analytically, especially when $N \to \infty$

To deal with this we re-express it in terms of a set of auxilliary functions:

Fusion hierarchy

$$T^{(j)}\left(u+\frac{\gamma}{2}\right)T^{(j)}\left(u-\frac{\gamma}{2}\right) = T^{(j-1)}(u)\,T^{(j+1)}(u) + \Phi_{j-1}(u) \qquad T^{(1)}(u) = T_{\text{open}}(u)$$

Y system

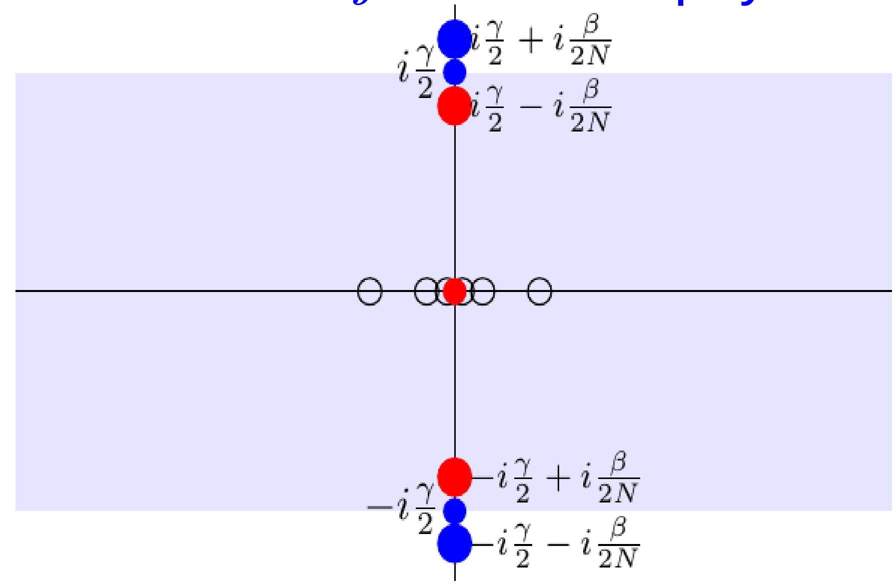
$$y_j\left(u+\frac{\gamma}{2}\right)y_j\left(u-\frac{\gamma}{2}\right) = [1+y_{j+1}(u)]\,[1+y_{j-1}(u)] \quad ★ \quad y_j(u) = \frac{T^{(j-1)}(u)T^{(j+1)}(u)}{\Phi_j(u)}$$

Relating Y functions to $\Lambda(u)$

$$1 + y_1(u) = \mathcal{N}(u)\,\Lambda\left(u+\frac{\gamma}{2}\right)\Lambda\left(u-\frac{\gamma}{2}\right) \quad ★★$$

Why all this works :

Eqs. ★ and ★★ can be recast as NLIE where the only dependence in Bethe roots and N is through the poles and zeroes of y_j inside the physical strip $\mathrm{Im}(iu) < \gamma/2$



- Much simpler than the Bethe roots
- N enters just as a parameter : straightforward Trotter limit !

★ →
$$\log[y_n(\lambda)] = d_n(\lambda) + [s * \{\log(1+y_{n-1}) + \log(1+y_{n+1})\}](\lambda)$$

Determined by the poles and zeroes

★★ →
$$\log\Lambda(\lambda) = \int_{-\pi/2}^{+\pi/2} d\mu\ s(\lambda-\mu)\left\{4\pi\beta a(\mu) + \log\left[\frac{1+y_1(\mu)}{1+Y_1(\lambda)}\right]\right\}$$

This system of equations can be solved numerically

For β real, all the solutions are real, the leading QTM eigenvalue remains the same with finite gap, etc… : all easy !

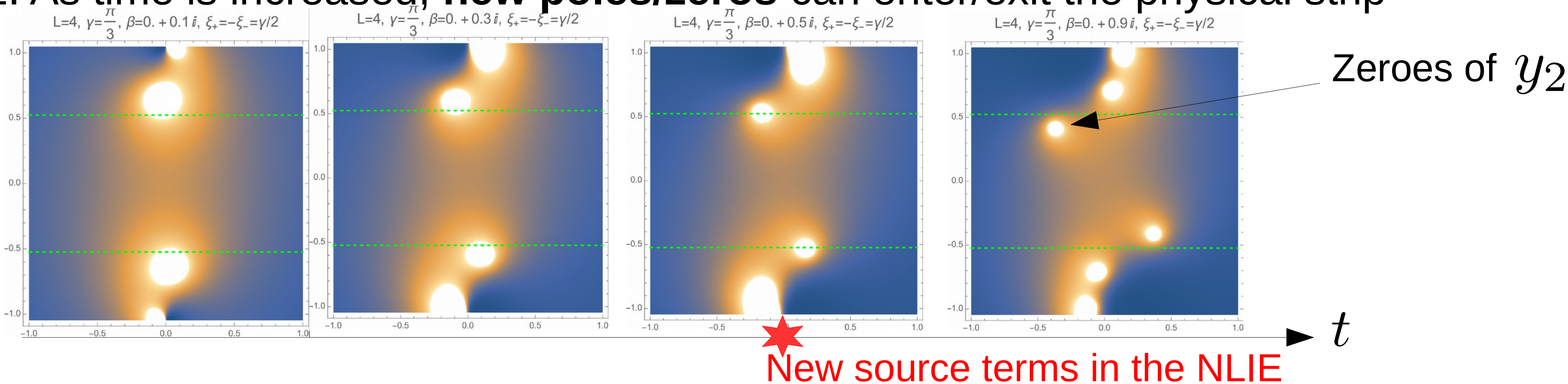
Continuing to real time (β imaginary), however, things become much richer

Continuation to real time

1. The solutions now take complex value, and a proper treatment requires solving the NLIE on a **Riemann surface**

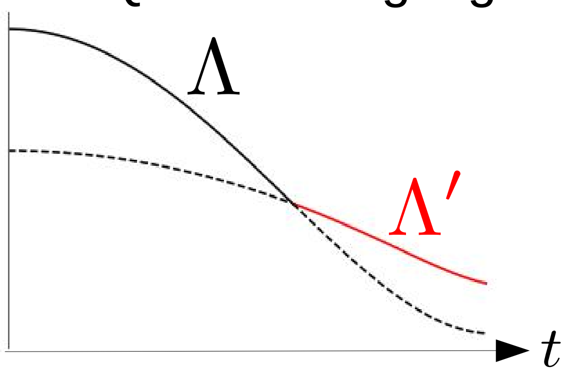
   Previous instances of “Riemann surface TBA” in the literature :
   Cavaglià, Fioravanti, Mattelliano, Tateo (2011)

2. As time is increased, **new poles/zeros** can enter/exit the physical strip



The NLIE with these new source terms are solved recursively for each t: exact location of the new poles determined self-consistently (“Excited state TBA”: **Dorey Tateo (1996)**)

3. The BQTM leading eigenvalue is now subject to **level crossings**

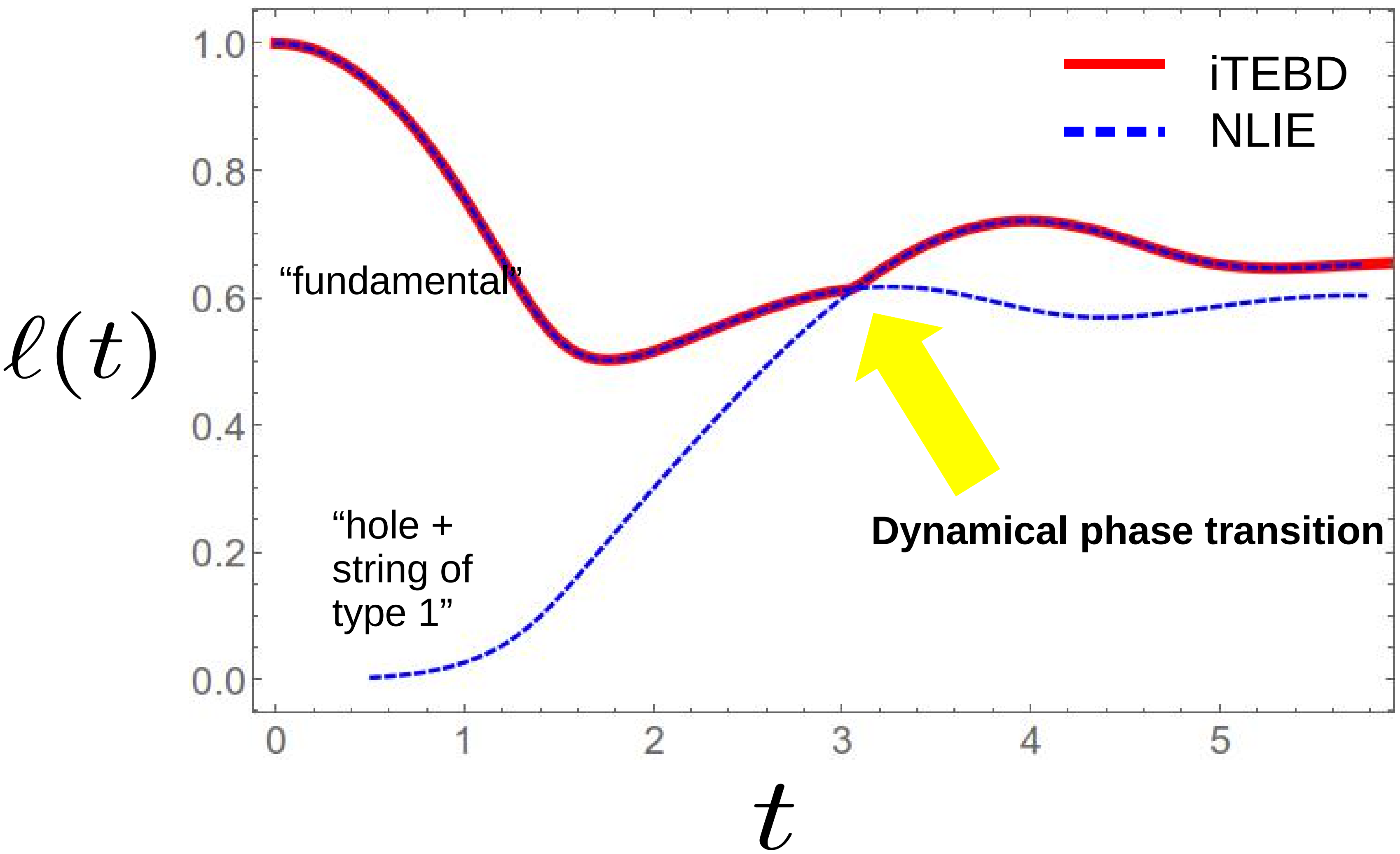


→ new excited levels

→ new zeroes and poles

→ new NLIE

Results



Our method allows in principle to compute $\ell(t)$ for arbitrarily large time, by solving the NLIE for all the relevant excitations (work in progress)

Summary so far : Exact calculation in XXZ starting from states of the form

$$|\Psi_0\rangle = |\psi_0\rangle_{1,2} \otimes |\psi_0\rangle_{3,4} \cdots \otimes |\psi_0\rangle_{L-1,L}$$

Other exact results for such states in the literature : closed form determinant formulas for the overlaps with Bethe states (Néel, dimer state,…)
Pozsgay (2014); Brockmann, De Nardis, Wouters & Caux (2014); Piroli & Calabrese (2014))

→ **What is special about states which allows for exact results ?**

Summary so far : Exact calculation in XXZ starting from states of the form

$$|\Psi_0\rangle = |\psi_0\rangle_{1,2} \otimes |\psi_0\rangle_{3,4} \cdots \otimes |\psi_0\rangle_{L-1,L}$$

Other exact results for such states in the literature : closed form determinant formulas for the overlaps with Bethe states (Néel, dimer state,…)
Pozsgay (2014); Brockmann, De Nardis, Wouters & Caux (2014); Piroli & Calabrese (2014))

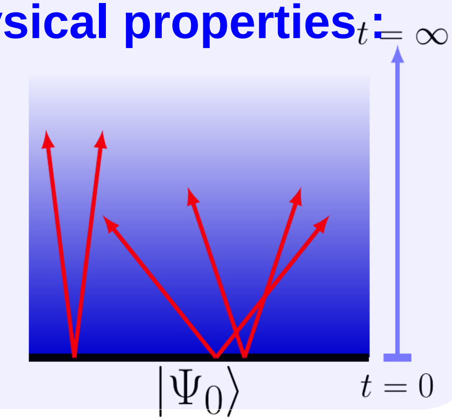
→ **What is special about states which allows for exact results ?**

Observation #1 : All these states share a number of interesting physical properties :

- Y system

- diagonal entropy = ½ Yang Yang entropy **Piroli, EV, Rigol, Calabrese**

- **pair structure :** overlap only with eigenstates corresponding to **symmetric** Bethe roots configurations



Summary so far : Exact calculation in XXZ starting from states of the form

$$|\Psi_0\rangle = |\psi_0\rangle_{1,2} \otimes |\psi_0\rangle_{3,4} \cdots \otimes |\psi_0\rangle_{L-1,L}$$

Other exact results for such states in the literature : closed form determinant formulas for the overlaps with Bethe states (Néel, dimer state,…)
Pozsgay (2014); Brockmann, De Nardis, Wouters & Caux (2014); Piroli & Calabrese (2014))

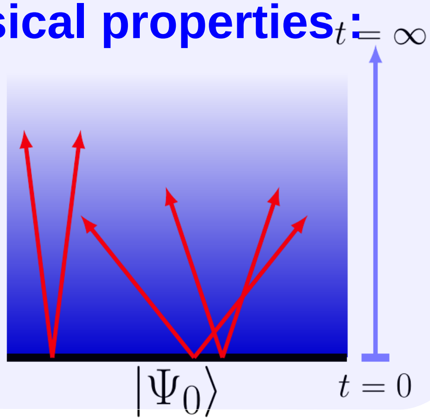
→ What is special about states which allows for exact results ?

Observation #1 : All these states share a number of interesting physical properties :

- Y system

- diagonal entropy = ½ Yang Yang entropy **Piroli, EV, Rigol, Calabrese**

- **pair structure :** overlap only with eigenstates corresponding to **symmetric** Bethe roots configurations



Observation #2 : this is not a specificity of 2-sites product states

In the AdS/CFT literature, exact overlap formulas for some classes of Matrix Product States for the SU(2) and SU(3) Heisenberg chains:

$$SU(2) \quad |\Psi_0^{(k)}\rangle = \mathrm{tr}_k \left[\prod_{l=1}^{L} \left(S_x^{(k)} |\uparrow\rangle_l + S_y^{(k)} |\downarrow\rangle_l \right) \right]$$

$$SU(3) \quad |\Psi_0\rangle = \mathrm{tr}_0 \left[\prod_{j=1}^{L} \left(\sigma_0^x |\Uparrow\rangle_j + \sigma_0^y |0\rangle_j + \sigma_0^z |\Downarrow\rangle_j \right) \right]$$

Determinant form of the overlaps, very reminiscent of known results for Néel etc...

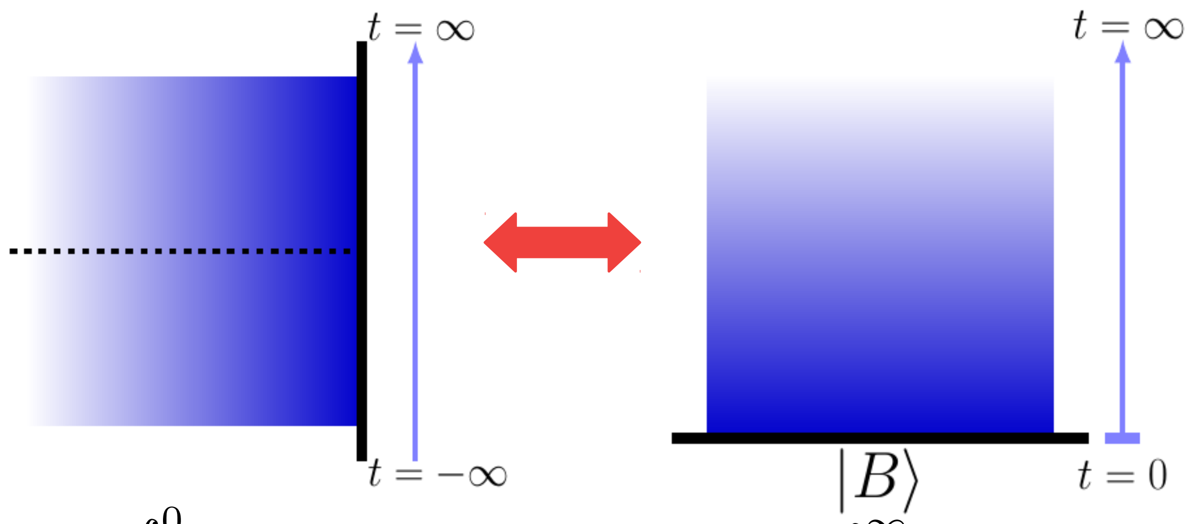
Buhl-Mortensen, de Leeuw, Kristjansen, Zarembo, (2015,2016)

...what is the underlying structure behind all this ?

class of “integrable initial states” ? How vast ?

“Integrable states” in field theory

Ghoshal Zamolodchikov (1992)



$$H_{\mathrm{B}} = \int_{-\infty}^{0} dx \; \hat{h}(x) + \theta_{\mathrm{B}} \qquad\qquad H = \int_{-\infty}^{\infty} dy \; \hat{h}(y)$$

Integrable bulk Hamiltonian :
Conserved charges $P_s \pm \bar{P}_s$

Integrable initial states $|B\rangle$ ⟷ boundary terms θ_{B} for which an infinity of conserved charges survive

Equivalent to :

$$\left(P_s - \bar{P}_s\right)|B\rangle = 0\,, \quad s \in \mathcal{S}_{\mathrm{B}}$$

$|B\rangle$ **annihilated by** (an infinite subset of) **the parity-odd bulk charges**

On the lattice things are a priori less clear: space discrete vs time continuous, no Lorentz invariance.

Nevertheless, we define the following :

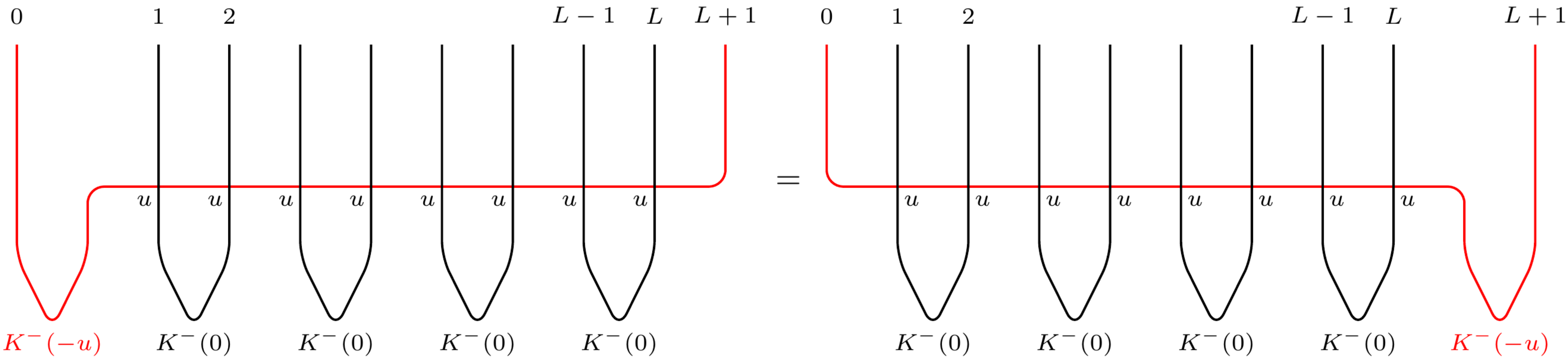
Lattice integrable states $\equiv$ **annihilated by all parity-odd charges**

$$Q_{2k+1}|\Psi_0\rangle = 0 \qquad k \geq 1$$

any system size

For 2-sites product states, the BQTM construction allows to relate this to the integrability of the corresponding reflection matrices

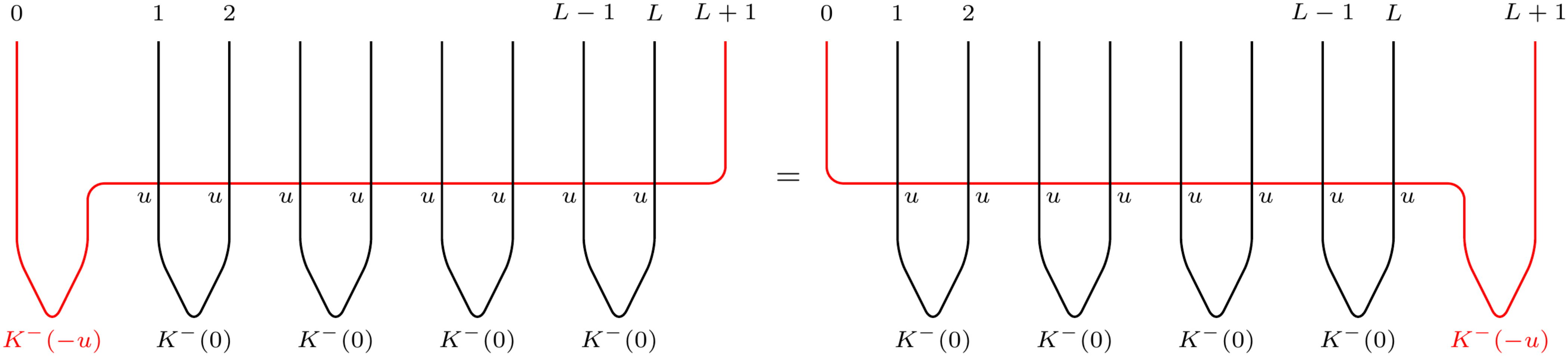
Sketch of the proof :



Then act on both sides with $K^{-}(-u)^{-1}$, then trace out the red space.
This proves $T(u)|\Psi_0\rangle = \Pi T(u)\Pi|\Psi_0\rangle$, so annihilation by all odd charges.

For 2-sites product states, the BQTM construction allows to relate this to the integrability of the corresponding reflection matrices

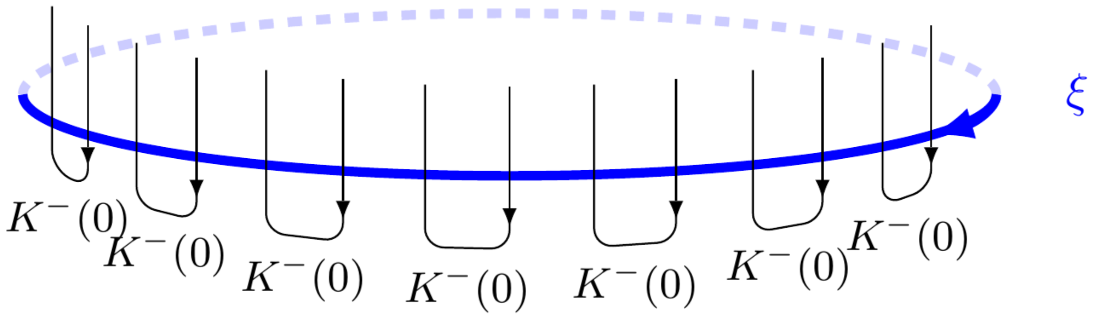
Sketch of the proof :



Then act on both sides with $K^-(-u)^{-1}$, then trace out the red space.
This proves $T(u)|\Psi_0\rangle = \Pi T(u) \Pi |\Psi_0\rangle$, so annihilation by all odd charges.

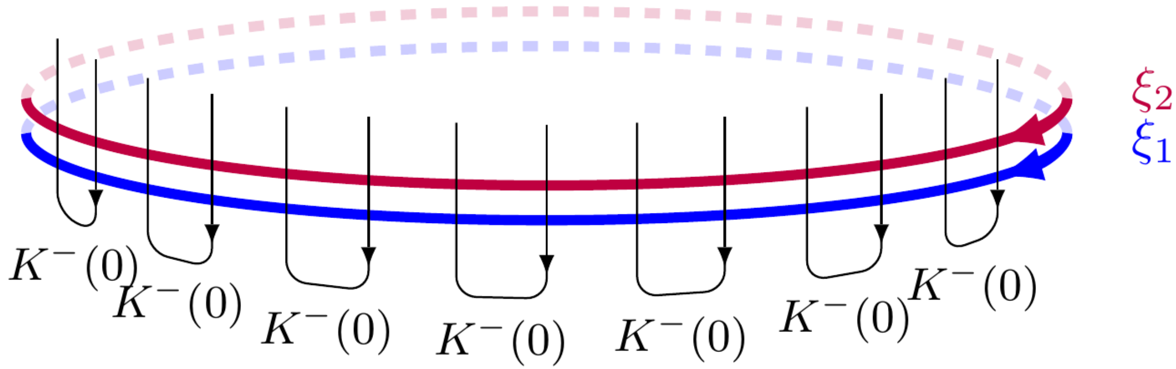
We can actually go further and interpret the previously mentioned MPS similarly

$$\left|\Psi_0^{\text{MPS}}\right\rangle = T(\xi)|\psi_0\rangle_{1,2} \otimes |\psi_0\rangle_{3,4} \cdots \otimes |\psi_0\rangle_{L-1,L}$$



- SU(2) AdS/CFT MPS are indeed of this form
- SU(3): under investigation

Even further : infinitely many possible integrable MPS, with arbitrary bond dimension



Act with arbitrary products of all possible kinds of transfer matrices
(arbitrary auxilliary representations, arbitrary spectral parameters)

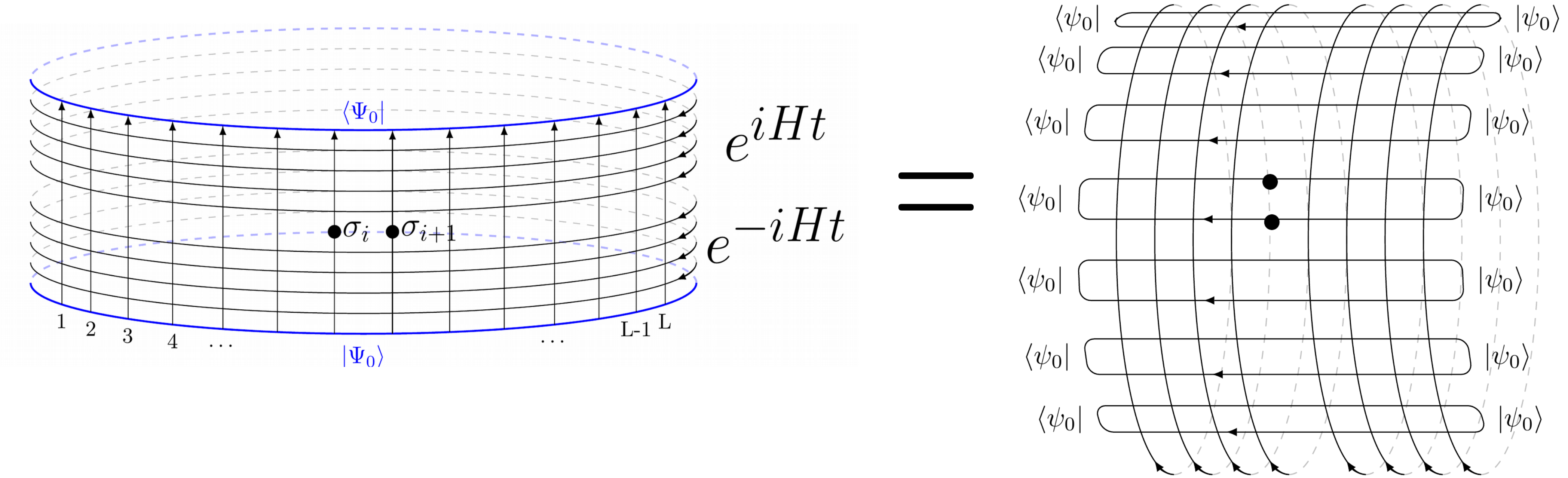
- for all such states we may expect closed-form determinant formulae for the overlaps

- can be used to approximate many physical states

- useful starting point in cases where little is known (ex: SU(N))

Outlook: further observables ?

Our framework can be naturally extended to the dynamics of local observables



(work in progress!)

Summary

- Method for **exactly computing the dynamics of observables** after a quantum quench

  In principle, gives access to **arbitrary times** (>> numerics) [in progress]

  Here we focused on Loschmidt echo : observation of **dynamical phase transitions**

- Classification of **“integrable initial states”** for which such a scheme is possible

  Infinite set of such states, including **MPS with arbitrarily large bond dimension**

  Useful starting point for models where quantum quenches have not yet been investigated (ex: SU(N))

Some (not so old) memories

She built him a trebuchet !

She built him a trebuchet !

Thank you for your attention,

and happy birthday Jean-Michel !