Toroidal symmetry in quantum integrable systems

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Introduction

- In this talk, we revisit an old topic in quantum integarble systems: the spectrum of integrals of motion in CFT.
- More specifically, we study a q-deformed version introduced by Feigin et al. 2007, from the point of view of quantum toroidal algebras.

Let us briefly recall what they are about.

The enveloping algebra of the Virasoro algebra contains well known elements (integrals of motion)

$$\mathbb{I}_1 = L_0 - \frac{c}{24}\,,$$
$$\mathbb{I}_3 = L_0^2 + 2\sum_{k=1}^{\infty} L_{-k}L_k - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}\,,$$
$$\mathbb{I}_5 = \sum_{k+l+m=0} :L_k L_l L_m: + \cdots\,,$$
$$\cdots\,,$$

which generate a commutative subalgebra $\mathcal{A} \subset (UVir)^{\wedge}$.
In particular, the $\mathbb{I}_j$'s define a commutative family of operators acting on each graded component of a Verma module, known also as quantum KdV system

Using integrals of screening currents, Bazhanov, Lukyanov and Zamolodchikov 1996 constructed another family of operators $\mathbb{G}_n$ $(n = 1, 2, \cdots)$ on the same Verma module. They satisfy

$$[\mathbb{I}_a, \mathbb{I}_b] = 0,\ [\mathbb{G}_k, \mathbb{G}_l] = 0,\ [\mathbb{I}_a, \mathbb{G}_k] = 0 \quad (\forall a, b, k, l).$$

The $\mathbb{I}_a$'s are called local IM, while the $\mathbb{G}_k$'s non-local IM.

A q-analog of local and non-local IM was found by Feigin, Kojima, Shiraishi and Watanabe 2007. They wrote formulas for IM in terms of the deformed Virasoro/screening currents, and proved their commutativity by hand.
The origin of these formulas has long remained unclear (to me).

Quantum toroidal algebras are quantization of loop algebras in two variables (Ginzburg et al. 1995)

affine	toroidal
$\mathfrak{gl}_n[x^{\pm 1}]$	$\mathfrak{gl}_n[x^{\pm 1}, y^{\pm 1}]$
$U_q\widehat{\mathfrak{gl}}_n$	$U_q\widehat{\widehat{\mathfrak{gl}}}_n$
$(\mathbb{C}^n)_u$	$\mathcal{F}_\nu(u)$

Main features of $\mathcal{E}_n = U_q\widehat{\widehat{\mathfrak{gl}}}_n$:

- 2 central elements $c, c^\perp$ (will choose $c^\perp = 0$)
- 2 deformation parameters, q_1,$q_2 = q^2$, q_3 with $q_1 q_2 q_3 = 1$
- Contains a Heisenberg algebra of n bosons
- Contains a quantum affine algebra $U_q\widehat{\mathfrak{gl}}_n$
- Presentation in terms of Drinfeld currents
- A Hopf algebra structure and a universal R matrix

Example.

$U_q\widehat{\widehat{\mathfrak{gl}}}_1$ has generators

$$e(z) = \sum_{n\in\mathbb{Z}} e_n z^{-n}, \quad f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n},$$

$$k^{\pm}(z) = q^{\mp c^{\perp}} \exp\Big(\sum_{\pm r>0} \kappa_r h_r z^{-r} \Big)\,,$$

$$\kappa_r = (1-q_1^r)(1-q_2^r)(1-q_3^r)\,.$$

Their defining relations are very similar to those of $U_q(\widehat{\mathfrak{sl}}_2)$, with the replacement of srtucture functions

$$z - q^2 w \quad \to \quad g(z,w) = (z-q_1 w)(z-q_2 w)(z-q_3 w)\,.$$

Relations ($C = q^c$, $k_0^- = q^{c^\perp}$):

$$k^\pm(z)k^\pm(w) = k^\pm(w)k^\pm(z),$$

$$\frac{g(C^{-1}z, w)}{g(Cz, w)} k^-(z)k^+(w) = \frac{g(w, C^{-1}z)}{g(w, Cz)} k^+(w)k^-(z),$$

$$g(z, w)k^\pm(C^{(-1\mp1)/2}z)e(w) + g(w, z)e(w)k^\pm(C^{(-1\mp1)/2}z) = 0,$$

$$g(w, z)k^\pm(C^{(-1\pm1)/2}z)f(w) + g(z, w)f(w)k^\pm(C^{(-1\pm1)/2}z) = 0\,,$$

$$[e(z), f(w)] = \frac{1}{\kappa_1}(\delta(\frac{Cw}{z})k^+(w) - \delta(\frac{Cz}{w})k^-(z)),$$

$$g(z, w)e(z)e(w) + g(w, z)e(w)e(z) = 0,$$

$$g(w, z)f(z)f(w) + g(z, w)f(w)f(z) = 0,$$

$$\mathop{\mathrm{Sym}}_{z_1,z_2,z_3} z_2 z_3^{-1}[e(z_1), [e(z_2), e(z_3)]] = 0\,,$$

$$\mathop{\mathrm{Sym}}_{z_1,z_2,z_3} z_2 z_3^{-1}[f(z_1), [f(z_2), f(z_3)]] = 0\,.$$

The most basic representations of $\mathcal{E}_n$ are the Fock modules

$$\mathcal{F}_\nu(u) = \mathcal{H} \otimes \bigoplus_{\beta \in \bar{Q} + \bar{\Lambda}_\nu} e^\beta \quad (\nu \in \mathbb{Z}/n\mathbb{Z})$$

where $\bar{Q}$ is the root lattice of $\widehat{\mathfrak{sl}}_n$ and

$$\mathcal{H} = \mathbb{C}[h_{i,-m} \mid m > 0, i \in \mathbb{Z}/n\mathbb{Z}]$$

is the bosonic Fock space of n bosons.

The currents are represented by vertex operators (Y.Saito 1998).
For example,

$$f_i(z) = u^{\delta_{i,0}} \exp\Big(\sum_{m>0} \frac{-1}{[m]} h_{i,-m} z^m\Big) \exp\Big(\sum_{m>0} \frac{q^m}{[m]} h_{i,m} z^{-m}\Big) e^{-\bar{\alpha}_i} z^{-h_{i,0}+1} .$$

Relation to CFT

Algebra $\mathcal{E}_n$ is interesting because of its close connection to CFT.

- The current $f(z)$ (or $e(z)$) on tensor products of $\mathcal{E}_1$ Fock modules are the deformed W_n currents enhanced by a Heisenberg algebra. (Miki 2007 $\cdots$)
- Deformed integrals of motion ('local' and 'non-local') are Taylor coefficients of transfer matrices.

Let us explain these points.

W_n currents

Consider the decomposition with respect to the Heisenberg subalgebra $\widehat{\mathfrak{gl}}_1$ of $\mathcal{E}_1$,

$$\mathcal{F}(v_1) \otimes \cdots \otimes \mathcal{F}(v_n) = \bigoplus M \otimes V \,.$$

Action of $f(z)$ decomposes into a product

$$\Delta^{(n-1)} f(z) = H(z) \otimes T(z)\,,$$

where $H(z)$ belongs to $\widehat{\mathfrak{gl}}_1$ and

$$T(z) = \sum_{j=1}^{n} u_j :\Lambda_j(z):$$

is the bosonized form of the basic deformed W_n current (Virasoro current for $n = 2$).

Transfer matrices

Algebra $\mathcal{E}_n$ has a Hopf structure; it is a Drinfeld double of a "Borel subalgebra" $\mathcal{B} \subset \mathcal{E}_n$, so it comes equipped with the universal R matrix

$$\mathcal{R} \quad \in \mathcal{B} \widehat{\otimes} \overline{\mathcal{B}} \subset \mathcal{E}_n \widehat{\otimes} \mathcal{E}_n \,.$$

We consider a weighted trace on the Fock space

$$T_\nu(u) = \mathrm{Tr}_{\mathcal{F}_\nu(u),1} \Big(\big(\bar{p}^d \prod_{i=1}^{n-1} \bar{p}_i^{\bar{\Lambda}_i} \big)_1 \mathcal{R}_{1,2} \Big) \,,$$

where $\bar{p}, \bar{p}_1, \ldots, \bar{p}_{n-1}$ are parameters, $\bar{\Lambda}_i$ are the fundamental weights of $\mathfrak{sl}_n$.

The coefficients in the expansion

$$T_\nu(u) = \sum_{\ell=0}^{\infty} I_{\nu,\ell}^{(n)} u^{-\ell}$$

are mutually commuting elements.

$I_{\nu,\ell}^{(n)}$ can be computed explicitly.
FJM 2017 (cf. Feigin-Tsymbaliuk 2015)

Remark. In the usual case of $U_q\widehat{\mathfrak{gl}}_2$, the XXZ Hamiltonian is obtained from a similar expansion, but at $u = 1$.

'Local' IM

Consider $\mathcal{E}_1$, and let $p = \bar{p} q^{-c}$. We have

$$I_\ell^{(1)} = c_\ell \int \cdots \int \overset{\curvearrowright}{\prod_{1 \le i \le \ell}} \hat{f}(x_i) \cdot \prod_{i<j} \frac{\Theta_p(x_j/x_i)\Theta_p(q_2 x_j/x_i)}{\Theta_p(q_1^{-1} x_j/x_i)\Theta_p(q_3^{-1} x_j/x_i)} \cdot \prod_{j=1}^{\ell} \frac{dx_j}{2\pi i x_j}\,,$$

$$\hat{f}(z) = f(z) \prod_{\ell=0}^{\infty} k^{+}((p^{\ell})^{-1} z)^{-1}\,,$$

$$\Theta_p(z) = \prod_{k=1}^{\infty} (1 - z p^{k-1})(1 - z^{-1} p^{k})(1 - p^{k}).$$

When acting on

$$W = \mathcal{F}(v_1) \otimes \cdots \otimes \mathcal{F}(v_n)\,,$$

$I_\ell^{(1)}$'s coincide with the (deformed) *local* IM of Feigin et al.

Duality and 'non-local' IM

There is a module $\mathbb{F}_{mn}$

$$
\begin{array}{ccccc}
\mathcal{E}_m(q_1, q_2, q_3) & & & & \mathcal{E}_n(q_1^\vee, q_2, q_3^\vee) \\
\bigcup & & & & \bigcup \\
U_q\widehat{\mathfrak{gl}}_m & \xrightarrow{\text{level } n} & \mathbb{F}_{mn} & \xleftarrow{\text{level } m} & U_q\widehat{\mathfrak{gl}}_n
\end{array}
$$

such that

- $[U_q\widehat{\mathfrak{gl}}_m, U_q\widehat{\mathfrak{gl}}_n] = 0$
- $[I^{(m)}_{\mu,\ell}(p), I^{(n)}_{\nu,\ell'}(p^\vee)] = 0$ provided $p = (q_1^\vee)^m, p^\vee = q_1^n$ (and appropriate relations between spectral parameters and $p_i, p_j^\vee$)

This is a toroidal analog of $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality due to Mukhin, Tarasov, Varchenko 2001,2005

In the case $m = 1$,

- $I_\ell^{(1)}(p)$: (deformed) 'local' IM
- $I_{\nu,\ell}^{(n)}(p^\vee)$: (deformed) 'non-local' IM

Bethe ansatz for $\mathfrak{gl}_1$ (Main result)

Let $W = \mathcal{F}(v_1) \otimes \cdots \otimes \mathcal{F}(v_n)$. For each eigenvector $w \in W$, there exists a polynomial $Q(u) = \prod_{i=1}^{N}(u - t_i)$ whose roots satisfy the Bethe ansatz equation

$$p \prod_{k=1}^{n} \frac{t_i - v_j}{t_i - q_2^{-1} v_k} \prod_{j=1}^{N} \prod_{s=1}^{3} \frac{q_s t_i - t_j}{q_s^{-1} t_i - t_j} = -1, \qquad i = 1, \dots, N\,.$$

The corresponding eigenvalues of IM are given by

$$I_1^{(1)} = C_1 \mathbf{w}_1\,, \quad I_2^{(1)} = C_2 \mathbf{w}_2 + \frac{1}{2!} C_{1,1} \mathbf{w}_1^2\,,$$
$$I_3^{(1)} = C_3 \mathbf{w}_3 + C_{2,1} \mathbf{w}_2 \mathbf{w}_1 + \frac{1}{3!} C_{1,1,1} \mathbf{w}_1^3\,, \cdots$$

where

$$\mathbf{w}_r = \sum_{j=1}^{N} t_j^r - \frac{q_3^r q_1^r}{(1 - q_3^r)(1 - q_1^r)} \sum_{k=1}^{n} v_k^r\,.$$

Outline of proof.

The usual algebraic Bethe ansatz seems hardly applicable. We use representation theory of "Borel subalgebra" $\mathcal{B}$ of $\mathcal{E}_1$.

(BLZ 1998;D.Hernandez and MJ 2013;E.Frenkel and D.Hernandez 2015)

- We construct $\mathcal{B}$-modules $M^+(u)$, $N^+(u)$ such that the following holds in the Grothendieck ring:

$$[N^+(u)][M^+(u)] = \prod_{s=1}^{3}[M^+(q_s^{-1}u)] + \prod_{s=1}^{3}[M^+(q_s u)]\{-1\}$$

NB: $k^+(z)$ has only one eigenvalue on $M^+(u)$.

- The corresponding (normalized) transfer matrices $Q(u)$, $\mathcal{T}(u)$ satisfy (warning: $\mathcal{T}(u) \neq T(u)$)

$$\mathcal{T}(u)Q(u) = a(u)\prod_{s=1}^{3} Q(q_s^{-1}u) + pd(u)\prod_{s=1}^{3} Q(q_s u)\,.$$

- With a suitable normalization, $Q(u)$, $\mathcal{T}(u)$ are polynomials on each $w \in W$. The roots of $Q(u)$ satisfy the Bethe equations

The eigenvalues of $T(u)$ are obtained by appropriate substitution into q-characters:

$$\frac{Q(u)}{Q(uq_2^{-1})}T(u) = \sum_{\lambda:\text{partitions}} \prod_{(i,j)\in\lambda} \mathfrak{a}(q_3^{-i}q_1^{-j}u)$$
$$= 1 + \mathfrak{a}(u) + \mathfrak{a}(u)\mathfrak{a}(q_3^{-1}u) + \mathfrak{a}(u)\mathfrak{a}(q_1^{-1}u) + \cdots\,,$$

where

$$\mathfrak{a}(u) = p\prod_{l=1}^{N} \frac{u - v_l}{qu - q^{-1}v_l}\prod_{s=1}^{3}\frac{Q(q_s u)}{Q(q_s^{-1}u)}\,.$$

We expect that the same method is applicable to $\mathfrak{gl}_n$ ($n \geq 2$).

Conjecture. The TQ relation holds in the form

$$\begin{aligned}\mathcal{T}_i(u)Q_i(u) = a_i(u)Q_{i-1}(q_1^{-1}u)Q_i(q_2^{-1}u)Q_{i+1}(q_3^{-1}u) \\ + p_i d_i(u)Q_{i-1}(q_3 u)Q_i(q_2 u)Q_{i+1}(q_1 u)\,,\end{aligned}$$

for $i = 0, 1, \ldots, n-1$.

The general case is technically more difficult. It is necessary to develop representation theory of quantum toroidal algebras (and of their Borel algebras).

ILW and Conformal limit

Consider the limit of $\mathcal{E}_1$ IM on $\mathcal{F}(v_1) \otimes \mathcal{F}(v_2)$:

$$q_1 = e^{-\beta h}, \quad q_2 = e^{(\beta-1)h}, \quad h \to 0,$$

keeping β and $p = e^{2\tau}$ fixed. We obtain

$$I_1^{(1)} = 2 + \beta h^2 \mathbf{I}_1 + \beta^{3/2} h^3 \mathbf{I}_2 + O(h^4),$$

$$\mathbf{I}_1 = L_0 + \frac{(1-\beta)^2}{4\beta} + 2 \sum_{m>0} a_{-m} a_m,$$

and the first Hamiltonian of Litvinov's ILW hierarchy

$$\mathbf{I}_2 = \sum_{m \neq 0} L_{-m} a_m - \frac{2(1-\beta)}{\sqrt{\beta}} \sum_{m>0} m \coth(m\tau) a_{-m} a_m + \frac{1}{3} \sum_{k+l+m=0} a_k a_l a_m$$

Our results give a proof of the Bethe equations conjectured by Litvinov 2013.

Summary

At present, there are 3 sets of Bethe equations which are known/expected to describe the spectrum of the same integrable system (IM in CFT):

$\mathfrak{gl}_1$ BA	local IM	proved
$\mathfrak{gl}_2$ BA	non-local IM	conjecture
BLZ BA	local IM	conjecture

The last one is given by the ODE/IM correspondence.

The most intriguing question: Where do the BLZ OPERs (Schrödinger operators) come from?

祝還暦

A Happy 60th Birthday!